Splitting a stationary set: Is there another way?

Arctic Set Theory Workshop 4,
Kilpisjärvi, 22-Jan-2019

Assaf Rinot
Bar-Ilan University, Israel

This talk is based on a joint work with Maxwell Levine.

Conventions

- κ denotes a regular uncountable cardinal;
- λ denotes an infinite cardinal;
- $\mathrm{Reg}(\kappa) := \{\lambda < \kappa \mid \aleph_0 \leq \mathrm{cf}(\lambda) = \lambda\}$;
- $E^\kappa_\lambda := \{\alpha < \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$;
- $E^\kappa_{\neq\lambda}$, $E^\kappa_{\geq\lambda}$ and $E^\kappa_{>\lambda}$ are defined analogously;
- $\mathrm{acc}^+(A) := \{\alpha < \sup(A) \mid \sup(A \cap \alpha) = \alpha > 0\}$.

Partitioning a stationary set

Theorem (Solovay, 1971)

For every stationary $S \subseteq \kappa$, there exists a partition $\langle S_i \mid i < \kappa \rangle$ of S into stationary sets.

Partitioning a stationary set

Theorem (Solovay, 1971)

For every stationary $S \subseteq \kappa$, there exists a partition $\langle S_i \mid i < \kappa \rangle$ of S into stationary sets.

Solovay's theorem has countless applications in Set Theory.
For instance, it plays a role in the proof of strong negative partition relations of the form $\kappa \nrightarrow [\kappa]^2_\kappa$, and variations of it are missing for the sought proof that successors of a singular cardinals cannot be Jónsson.

Partitioning a stationary set

Theorem (Solovay, 1971)

For every stationary $S \subseteq \kappa$, there exists a partition $\langle S_i \mid i < \kappa \rangle$ of S into stationary sets.

Solovay's theorem has countless applications in Set Theory.

You

What is your favorite application?

Variations of Solovay's theorem

Variation I (Brodsky-Rinot, 2019)

For every $\theta \leq \kappa$ and a sequence $\langle S_i \mid i < \theta \rangle$ of stationary subsets of κ, there exists a cofinal $I \subseteq \theta$ and pairwise disjoint stationary sets $\langle T_i \mid i \in I \rangle$ such that $T_i \subseteq S_i$ for all $i \in I$.

Variations of Solovay's theorem

Variation I (Brodsky-Rinot, 2019)

For every $\theta \leq \kappa$ and a sequence $\langle S_i \mid i < \theta \rangle$ of stationary subsets of κ, there exists a cofinal $I \subseteq \theta$ and pairwise disjoint stationary sets $\langle T_i \mid i \in I \rangle$ such that $T_i \subseteq S_i$ for all $i \in I$.

Variation II (Magidor?, 1970's)

If $\square_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$, there is a partition $\langle S_i \mid i < \lambda^+ \rangle$ of S into stationary sets such that, for all $i < \lambda^+$, S_i does not reflect.

Variations of Solovay's theorem

Definition

For $S \subseteq \kappa$, let $\mathrm{Tr}(S) := \{\beta \in E^{\kappa}_{>\omega} \mid S \cap \beta \text{ is stationary in } \beta\}$.

Variation II (Magidor?, 1970's)

If $\square_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$, there is a partition $\langle S_i \mid i < \lambda^+ \rangle$ of S into stationary sets such that, for all $i < \lambda^+$, S_i does not reflect (i.e., $\mathrm{Tr}(S_i) = \emptyset$).

Variations of Solovay's theorem

Variation II (Magidor?, 1970's)

If $\square_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$, there is a partition $\langle S_i \mid i < \lambda^+ \rangle$ of S into stationary sets such that, for all $i < \lambda^+$, S_i does not reflect (i.e., $\mathrm{Tr}(S_i) = \emptyset$).

$\looparrowright$ Nonreflecting stationary sets are very useful. To exemplify:

Variations of Solovay's theorem

Variation II (Magidor?, 1970's)

If $\square_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$, there is a partition $\langle S_i \mid i < \lambda^+ \rangle$ of S into stationary sets such that, for all $i < \lambda^+$, S_i does not reflect (i.e., $\mathrm{Tr}(S_i) = \emptyset$).

$\looparrowright$ Nonreflecting stationary sets are very useful. To exemplify:

Theorem (Shelah, 1991)

If $\kappa > \aleph_2$, and $E^\kappa_{\geq \aleph_2}$ admits a nonreflecting stationary set, then there exists a κ-cc poset whose square is not κ-cc.

Variations of Solovay's theorem

Variation III (Brodsky-Rinot, 2019)

If $\square(\kappa)$ holds, then for every fat $F \subseteq \kappa$, there is a partition $\langle F_i \mid i < \kappa \rangle$ of F into fat sets such that, for all $i < j < \kappa$, $\mathrm{Tr}(F_i) \cap \mathrm{Tr}(F_j) = \emptyset$.

Variation II (Magidor?, 1970's)

If $\square_\lambda$ holds, then for every stationary $S \subseteq \lambda^+$, there is a partition $\langle S_i \mid i < \lambda^+ \rangle$ of S into stationary sets such that, for all $i < \lambda^+$, S_i does not reflect (i.e., $\mathrm{Tr}(S_i) = \emptyset$).

$\looparrowright$ Nonreflecting stationary sets are very useful. To exemplify:

Theorem (Shelah, 1991)

If $\kappa > \aleph_2$, and $E^\kappa_{\geq \aleph_2}$ admits a nonreflecting stationary set, then there exists a κ-cc poset whose square is not κ-cc.

Variations of Solovay's theorem

Variation III (Brodsky-Rinot, 2019)

If $\square(\kappa)$ holds, then for every fat $F \subseteq \kappa$, there is a partition $\langle F_i \mid i < \kappa \rangle$ of F into fat sets such that, for all $i < j < \kappa$, $\mathrm{Tr}(F_i) \cap \mathrm{Tr}(F_j) = \emptyset$.

↬ Partitions as above are sometime enough:

Theorem (Rinot, 2014)

If $\kappa \geq \aleph_2$, and $\square(\kappa)$ holds,
then there exists a κ-cc poset whose square is not κ-cc.

↬ Nonreflecting stationary sets are very useful. To exemplify:

Theorem (Shelah, 1991)

If $\kappa > \aleph_2$, and $E^{\kappa}_{\geq \aleph_2}$ admits a nonreflecting stationary set,
then there exists a κ-cc poset whose square is not κ-cc.

Is there another way?

As said, partitioning κ into stationary sets that pairwise do not simultaneously reflect is very useful, but is also somewhat wired into the standard procedure of the partition.

Is there another way?

As said, partitioning κ into stationary sets that pairwise do not simultaneously reflect is very useful, but is also somewhat wired into the standard procedure of the partition.

Questions

- Is it possible to partition κ into two reflecting stationary sets?

Is there another way?

As said, partitioning κ into stationary sets that pairwise do not simultaneously reflect is very useful, but is also somewhat wired into the standard procedure of the partition.

Questions

- Is it possible to partition κ into two reflecting stationary sets?
- Is it possible to partition κ into κ reflecting stationary sets?

Is there another way?

As said, partitioning κ into stationary sets that pairwise do not simultaneously reflect is very useful, but is also somewhat wired into the standard procedure of the partition.

Questions

- Is it possible to partition κ into two reflecting stationary sets?
- Is it possible to partition κ into κ reflecting stationary sets?
- Is it possible to partition κ into $\langle S_i \mid i < \kappa \rangle$ such that, for all $i < j < \kappa$, $\mathrm{Tr}(S_i) \cap \mathrm{Tr}(S_j)$ be stationary?

Is there another way?

As said, partitioning κ into stationary sets that pairwise do not simultaneously reflect is very useful, but is also somewhat wired into the standard procedure of the partition.

Questions

- Is it possible to partition κ into two reflecting stationary sets?
- Is it possible to partition κ into κ reflecting stationary sets?
- Is it possible to partition κ into $\langle S_i \mid i < \kappa \rangle$ such that, for all $i < j < \kappa$, $\mathrm{Tr}(S_i) \cap \mathrm{Tr}(S_j)$ be stationary?
- Is it possible to partition κ into $\langle S_i \mid i < \kappa \rangle$ such that $\bigcap_{i<\kappa} \mathrm{Tr}(S_i)$ be stationary?

Is there another way?

As said, partitioning κ into stationary sets that pairwise do not simultaneously reflect is very useful, but is also somewhat wired into the standard procedure of the partition.

Questions

- Is it possible to partition κ into two reflecting stationary sets?
- Is it possible to partition κ into κ reflecting stationary sets?
- Is it possible to partition κ into $\langle S_i \mid i < \kappa\rangle$ such that, for all $i < j < \kappa$, $\mathrm{Tr}(S_i) \cap \mathrm{Tr}(S_j)$ be stationary?
- Is it possible to partition κ into $\langle S_i \mid i < \kappa\rangle$ such that $\bigcap_{i<\kappa} \mathrm{Tr}(S_i)$ be stationary?

Definition

$\Pi(S, \theta)$ asserts the existence of a partition $\langle S_i \mid i < \theta\rangle$ of S such that $\bigcap_{i<\theta} \mathrm{Tr}(S_i)$ is stationary.

Is there another way?

As said, partitioning κ into stationary sets that pairwise do not simultaneously reflect is very useful, but is also somewhat wired into the standard procedure of the partition.

Questions

- Is it possible to partition κ into two reflecting stationary sets?
- Is it possible to partition κ into κ reflecting stationary sets?
- Is it possible to partition κ into $\langle S_i \mid i < \kappa \rangle$ such that, for all $i < j < \kappa$, $\mathrm{Tr}(S_i) \cap \mathrm{Tr}(S_j)$ be stationary?
- Is it possible to partition κ into $\langle S_i \mid i < \kappa \rangle$ such that $\bigcap_{i<\kappa} \mathrm{Tr}(S_i)$ be stationary?

Definition

$\Pi(S, \theta, T)$ asserts the existence of a partition $\langle S_i \mid i < \theta \rangle$ of S such that $\bigcap_{i<\theta} \mathrm{Tr}(S_i) \cap T$ is stationary.

Singular cardinals combinatorics

Scales

Definition

Suppose that λ is a singular cardinal, and $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ is a strictly increasing sequence of regular cardinals, converging to λ. For any two functions $f, g \in \prod \vec{\lambda}$ and $i < \mathrm{cf}(\lambda)$, we write $f <^i g$ to express that $f(j) < g(j)$ whenever $i \leq j < \mathrm{cf}(\lambda)$.

Scales

Definition

Suppose that λ is a singular cardinal, and $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ is a strictly increasing sequence of regular cardinals, converging to λ. For any two functions $f, g \in \prod \vec{\lambda}$ and $i < \mathrm{cf}(\lambda)$, we write $f <^i g$ to express that $f(j) < g(j)$ whenever $i \leq j < \mathrm{cf}(\lambda)$.
We write $f <^* g$ to express that $f <^i g$ for some $i < \mathrm{cf}(\lambda)$.

Scales

Definition

Suppose that λ is a singular cardinal, and $\vec{\lambda} = \langle \lambda_i \mid i < \mathrm{cf}(\lambda) \rangle$ is a strictly increasing sequence of regular cardinals, converging to λ.
For any two functions $f, g \in \prod \vec{\lambda}$ and $i < \mathrm{cf}(\lambda)$, we write $f <^i g$ to express that $f(j) < g(j)$ whenever $i \leq j < \mathrm{cf}(\lambda)$.
We write $f <^* g$ to express that $f <^i g$ for some $i < \mathrm{cf}(\lambda)$.

Definition

Suppose that λ is a singular cardinal; $\vec{f} = \langle f_\beta \mid \beta < \lambda^+ \rangle$ is said to be a scale for λ iff there exists a sequence $\vec{\lambda}$ as above, such that:

- for every $\beta < \lambda^+$, $f_\beta \in \prod \vec{\lambda}$;
- for every $\beta < \alpha < \lambda^+$, $f_\beta <^* f_\alpha$;
- for every $g \in \prod \vec{\lambda}$, there exists $\beta < \lambda^+$ such that $g <^* f_\beta$.

Scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a scale.

Definition

Suppose that λ is a singular cardinal; $\vec{f} = \langle f_\beta \mid \beta < \lambda^+ \rangle$ is said to be a scale for λ iff there exists a sequence $\vec{\lambda}$ as above, such that:

- for every $\beta < \lambda^+$, $f_\beta \in \prod \vec{\lambda}$;
- for every $\beta < \alpha < \lambda^+$, $f_\beta <^* f_\alpha$;
- for every $g \in \prod \vec{\lambda}$, there exists $\beta < \lambda^+$ such that $g <^* f_\beta$.

Scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \vec{\lambda}$.
An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be good if there exist $i < \mathrm{cf}(\lambda)$ and a cofinal $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.

Definition

Suppose that λ is a singular cardinal; $\vec{f} = \langle f_\beta \mid \beta < \lambda^+ \rangle$ is said to be a scale for λ iff there exists a sequence $\vec{\lambda}$ as above, such that:

- for every $\beta < \lambda^+$, $f_\beta \in \prod \vec{\lambda}$;
- for every $\beta < \alpha < \lambda^+$, $f_\beta <^* f_\alpha$;
- for every $g \in \prod \vec{\lambda}$, there exists $\beta < \lambda^+$ such that $g <^* f_\beta$.

Scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \vec{\lambda}$.
An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be good if there exist $i < \mathrm{cf}(\lambda)$ and a cofinal $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.
We let $G(\vec{f})$ denote the set of good points with respect to $\vec{f}$.

Definition

Suppose that λ is a singular cardinal; $\vec{f} = \langle f_\beta \mid \beta < \lambda^+ \rangle$ is said to be a scale for λ iff there exists a sequence $\vec{\lambda}$ as above, such that:

- for every $\beta < \lambda^+$, $f_\beta \in \prod \vec{\lambda}$;
- for every $\beta < \alpha < \lambda^+$, $f_\beta <^* f_\alpha$;
- for every $g \in \prod \vec{\lambda}$, there exists $\beta < \lambda^+$ such that $g <^* f_\beta$.

Scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \vec{\lambda}$.
An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be good if there exist $i < \mathrm{cf}(\lambda)$ and a cofinal $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.
We let $G(\vec{f})$ denote the set of good points with respect to $\vec{f}$.

The set of good points is stationary (Shelah, 1990's)

For every regular θ with $\mathrm{cf}(\lambda) < \theta < \lambda$, $G(\vec{f}) \cap E^{\lambda^+}_\theta$ is stationary.

Scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \vec{\lambda}$.
An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be good if there exist $i < \mathrm{cf}(\lambda)$ and a cofinal $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.
We let $G(\vec{f})$ denote the set of good points with respect to $\vec{f}$.

The set of good points is stationary (Shelah, 1990's)

For every regular θ with $\mathrm{cf}(\lambda) < \theta < \lambda$, $G(\vec{f}) \cap E^{\lambda^+}_\theta$ is stationary.

The set of good points is robust

If $\vec{f}, \vec{g}$ are scales in $\prod \vec{\lambda}$, then $G(\vec{f}) \bigtriangleup G(\vec{g})$ is nonstationary.

Scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \vec{\lambda}$.
An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be very good if there exist $i < \mathrm{cf}(\lambda)$ and a ~~cofinal~~ club $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.

Scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \vec{\lambda}$.
An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be very good if there exist $i < \mathrm{cf}(\lambda)$ and a ~~cofinal~~ club $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.
We let $V(\vec{f})$ denote the set of very good points with respect to $\vec{f}$.

Scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \vec{\lambda}$.
An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be very good if there exist $i < \mathrm{cf}(\lambda)$ and a ~~cofinal~~ club $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.
We let $V(\vec{f})$ denote the set of very good points with respect to $\vec{f}$.

Recall

If $\vec{f}, \vec{g}$ are scales in $\prod \vec{\lambda}$, then $G(\vec{f}) \triangle G(\vec{g})$ is nonstationary.

Scales

Theorem (Shelah, 1990's)

Every singular cardinal λ admits a scale.

Suppose $\vec{f}$ is a scale in $\prod \vec{\lambda}$.
An ordinal $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ is said to be very good if there exist $i < \mathrm{cf}(\lambda)$ and a ~~cofinal~~ club $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^i f_\gamma$.
We let $V(\vec{f})$ denote the set of very good points with respect to $\vec{f}$.

Recall

If $\vec{f}, \vec{g}$ are scales in $\prod \vec{\lambda}$, then $G(\vec{f}) \triangle G(\vec{g})$ is nonstationary.

Theorem (Cummings-Foreman, 2010)

If $V = L$, then there are scales $\vec{f}, \vec{g}$ in $\prod_{n<\omega} \aleph_n$ for which $V(\vec{f}) = E^{\aleph_{\omega+1}}_{>\omega}$ and $V(\vec{g}) = \emptyset$.

Very good points are not robust

The following is implicit in the proof of the above-mentioned theorem of Cummings-Foreman concerning $V = L$:

Proposition

Suppose λ is singular, $T \subseteq \lambda^+$ is stationary and $\Pi(\lambda^+, \mathrm{cf}(\lambda), T)$. Suppose $\vec{f}$ is a scale for λ, living in some product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$. Then $T \setminus V(\vec{g})$ is stationary for some scale $\vec{g}$ in $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$.

Proof.

Fix a partition $\langle S_i \mid i < \mathrm{cf}(\lambda)\rangle$ of λ^+, with $T' := T \cap \bigcap_{i<\mathrm{cf}(\lambda)} \mathrm{Tr}(S_i)$ stationary. Define $\langle g_\beta \mid \beta < \lambda^+\rangle$ by letting $g_\beta(i) := 0$ for $\beta \in S_i$, and $g_\beta(i) := f_\beta(i)$, otherwise.
Let $\alpha \in T'$ be arbitrary. To see that $\alpha \notin V(\vec{g})$, fix an arbitrary club $C \subseteq \alpha$ and an index $i < \mathrm{cf}(\lambda)$.
Let $\delta := \min(C \cap S_i)$ and $\gamma := \min(C \cap S_i \setminus (\delta + 1))$.
Then $\delta < \gamma$ is a pair of elements of C, while $g_\delta(i) = 0 = g_\gamma(i)$. □

Very good scales

Definition

A scale $\vec{f}$ for a singular cardinal λ is said to be very good iff club many $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ are very good for $\vec{f}$.

Very good scales

Definition

A scale $\vec{f}$ for a singular cardinal λ is said to be very good iff club many $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ are very good for $\vec{f}$.

Conclusion

Suppose λ is a singular cardinal and $\Pi(\lambda^+, \mathrm{cf}(\lambda), E^{\lambda^+}_{>\mathrm{cf}(\lambda)})$ holds. Then any product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$ admitting a scale for λ, admits yet another scale which is not very good.

Very good scales

Definition

A scale $\vec{f}$ for a singular cardinal λ is said to be very good iff club many $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ are very good for $\vec{f}$.

Conclusion

Suppose λ is a singular cardinal and $\Pi(\lambda^+, \mathrm{cf}(\lambda), E^{\lambda^+}_{>\mathrm{cf}(\lambda)})$ holds. Then any product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$ admitting a scale for λ, admits yet another scale which is not very good.

Note

There are numerous ways to consistently get instances of $\Pi(S, \theta, T)$. For instance, in a model of Magidor (1982), $\Pi(S, \aleph_1, E^{\aleph_2}_{\aleph_1})$ holds for every stationary $S \subseteq E^{\aleph_2}_{\aleph_0}$.
The main point here is to prove instances of $\Pi(S, \theta, T)$ in ZFC.

ZFC results

Main result

Theorem

Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. *If λ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;*

This is trivial

Simply take $\langle E^{\lambda}_{\mu} \mid \mu \in \mathrm{Reg}(\aleph_{\theta+1}) \rangle$ and $\langle E^{\lambda^+}_{\mu} \mid \mu \in \mathrm{Reg}(\lambda) \rangle$.

Main result

Theorem

Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. *If λ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;*
2. *If λ is regular, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*

This is optimal

If $\Pi(S, \theta, T)$ holds, then $\{\alpha \in T \mid \mathrm{cf}(\alpha) \geq \theta\}$ must be stationary.

Main result

Theorem

Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. *If λ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;*
2. *If λ is regular, then $\Pi(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_\theta)$ holds;*
3. *If $2^\theta \leq \lambda$ and $\theta \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_\mu, \theta, E^{\lambda^+}_\theta)$ holds;*

Main result

Theorem

Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. *If λ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;*
2. *If λ is regular, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*
3. *If $2^\theta \leq \lambda$ and $\theta \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*
4. *If λ is singular and $\theta^{++} \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{++}})$ holds;*

Main result

Theorem

Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. *If λ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;*
2. *If λ is regular, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*
3. *If $2^\theta \leq \lambda$ and $\theta \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*
4. *If λ is singular and $\theta^{++} \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{++}})$ holds;*
5. *If λ is singular and $\theta^{++} = \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{+3}})$ holds.*

Remark

This follows from Clause (4).

Main result

Theorem

Suppose that $\mu < \theta$ are infinite regular cardinals $< \lambda$.

1. *If λ is inaccessible, then $\Pi(\lambda, \theta, \lambda)$ and $\Pi(\lambda^+, \lambda, \lambda^+)$ hold;*
2. *If λ is regular, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*
3. *If $2^\theta \leq \lambda$ and $\theta \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta})$ holds;*
4. *If λ is singular and $\theta^{++} \neq \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{++}})$ holds;*
5. *If λ is singular and $\theta^{++} = \mathrm{cf}(\lambda)$, then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{+3}})$ holds.*

Remark

Our proof at the level of successors of singulars is indeed different from the standard proofs for partitioning a stationary set. We build on the fact that any singular cardinal admits a scale and that the set of good points of a scale is stationary relative to any cofinality; we also use a combination of Ulam matrices with club-guessing to avoid any cardinal arithmetic hypotheses (Clauses (4) and (5)).

A special case with a simplified proof

Theorem

Let λ be a singular cardinal. Let $\mu < \theta$ be regular cardinals with $\mathrm{cf}(\lambda) < \mu < \theta < \lambda$. *Then* $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{++}})$ *holds.*

A special case with a simplified proof

Theorem

Let λ be a singular cardinal. Let $\mu < \theta$ be regular cardinals with $\mathrm{cf}(\lambda) < \mu < \theta < \lambda$. Then $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{++}})$ holds.

Proof. Fix a scale $\vec{f}$ for λ in some product $\prod_{i<\mathrm{cf}(\lambda)} \lambda_i$.

By Shelah's theorem, $T_0 := E^{\lambda^+}_{\theta^{++}} \cap G(\vec{f})$ is stationary.

Claim 1

There exist $i < \mathrm{cf}(\lambda)$, $\zeta \in E^{\lambda}_{\theta^{++}}$, a stationary $T_1 \subseteq T_0$, and a sequence $\langle S^1_\alpha \mid \alpha \in T_1 \rangle$ such that, for all $\alpha \in T_1$:

- *S^1_α is a stationary subset of E^{α}_{μ};*
- *$\langle f_\beta(i) \mid \beta \in S^1_\alpha \rangle$ is strictly increasing and converging to ζ.*

Proof. By Fodor's lemma, it suffices to prove that for each $\alpha \in T_0$, there is $i < \mathrm{cf}(\lambda)$ and a stationary $S \subseteq E^{\alpha}_{\mu}$ on which $\beta \mapsto f_\beta(i)$ is strictly increasing.

Proof of Claim 1

Let $\alpha \in T_0$ be arbitrary. We shall find $i < \mathrm{cf}(\lambda)$ and a stationary $S \subseteq E^\alpha_\mu$ on which $\beta \mapsto f_\beta(i)$ is strictly increasing.
For each $\gamma \leq \beta < \alpha$, pick $i_{\gamma,\beta} < \mathrm{cf}(\lambda)$ such that $f_\gamma <^{i_{\gamma,\beta}} f_\beta$.
As $\alpha \in T_0$ is a good point, let us also fix $i' < \mathrm{cf}(\lambda)$ and a cofinal $A \subseteq \alpha$ such that, for all $\delta < \gamma$ from A, $f_\delta <^{i'} f_\gamma$.
Consider $S' := \mathrm{acc}^+(A) \cap E^\alpha_\mu$, which is a stationary subset of E^α_μ.
As $\mu > \mathrm{cf}(\lambda)$, for each $\beta \in S'$, we may pick a cofinal $a_\beta \subseteq A \cap \beta$ and $i_\beta < \mathrm{cf}(\lambda)$ such that, for all $\gamma \in a_\beta$, $i_{\gamma,\beta} = i_\beta$.
As $\theta^{++} > \mathrm{cf}(\lambda)$, we may pick a stationary $S \subseteq S'$ and $i < \mathrm{cf}(\lambda)$ such that, for all $\beta \in S$, $\max\{i_\beta, i', i_{\beta,\min(A \setminus \beta)}\} = i$.
To see that i and S are as sought, let $\epsilon < \beta$ be arbitrary elements of S. Consider $\delta := \min(A \setminus \epsilon)$ and $\gamma := \min(a_\beta \setminus \delta)$.
Clearly, $\epsilon \leq \delta \leq \gamma < \beta$ and $f_\epsilon <^{i_{\epsilon,\min(A\setminus\epsilon)}} f_\delta <^{i'} f_\gamma <^{i_\beta} f_\beta$.
In particular, $f_\epsilon <^i f_\beta$, so that $f_\epsilon(i) < f_\beta(i)$, as sought. □

Fix i, ζ, and $\langle S^1_\alpha \mid \alpha \in T_1 \rangle$ as in Claim 1.

Step 2: Find a function g

Claim 2

There are $g : E^{\lambda^+}_\mu \to \theta^{++}$ and a sequence $\langle S^2_\alpha \mid \alpha \in T_1\rangle$ such that, for all $\alpha \in T_1$:

- *S^2_α is a stationary subset of S^1_α (hence, of E^α_μ);*
- *$\langle g(\beta) \mid \beta \in S^2_\alpha\rangle$ is strictly increasing (hence, cofinal in θ^{++}).*

Proof. Fix a club z in ζ with $\mathrm{otp}(z) = \theta^{++}$. Define $g : E^{\lambda^+}_\mu \to \theta^{++}$ by letting $g(\beta) := \mathrm{otp}(f_\beta(i) \cap z)$ if $f_\beta(i) < \zeta$ and $g(\beta) := 0$, o.w. To see that g is as sought, let $\alpha \in T_1$ be arbitrary. Let $\pi : \theta^{++} \to \alpha$ be the inverse collapse of some club in α. Clearly, $\bar{S} := \{\bar{\beta} < \theta^{++} \mid \pi(\bar{\beta}) \in S^1_\alpha \;\&\; (g \circ \pi)``\bar{\beta} \subseteq \bar{\beta}\}$ is stationary. Let $\bar{B} := \{\bar{\beta} \in \bar{S} \mid (g \circ \pi)(\bar{\beta}) < \bar{\beta}\}$. For all $\bar{\epsilon} < \bar{\beta}$ from $\bar{S} \setminus \bar{B}$, we have $g(\pi(\bar{\epsilon})) < \bar{\beta} \leq g(\pi(\bar{\beta}))$. Thus, it suffices to show that $S^2_\alpha := \pi[\bar{S} \setminus \bar{B}]$ (which is a subset of S^1_α) is stationary. Suppose not. In particular, $\bar{B}$ is stationary. But then, Fodor's lemma entails a stationary $\hat{B} \subseteq \bar{B}$ on which $g \circ \pi$ is constant, contradicting the fact that $\langle f_{\pi(\bar{\beta})}(i) \mid \bar{\beta} \in \hat{B}\rangle$ converges to ζ. □

Step 3: An Ulam Matrix

Let $g : E^{\lambda^+}_\mu \to \theta^{++}$ and $\langle S^2_\alpha \mid \alpha \in T_1\rangle$ be given by Claim 2.
Now, fix an Ulam matrix $\langle A_{\xi,\eta} \mid \xi < \theta^{++}, \eta < \theta^+\rangle$ over θ^{++}, i.e.,

- for all $\xi < \theta^{++}$, $|\theta^{++} \setminus \bigcup_{\eta<\theta^+} A_{\xi,\eta}| \leq \theta^+$;
- for all $\eta < \theta^+$ and $\xi < \xi' < \theta^{++}$, $A_{\xi,\eta} \cap A_{\xi',\eta} = \emptyset$.

Claim 3

For every $\alpha \in T_1$, there are $\eta < \theta^+$ and $x \in [\theta^{++}]^{\theta^{++}}$ such that, for all $\xi \in x$, $g^{-1}[A_{\xi,\eta}] \cap \alpha$ is stationary in α.

Proof. Suppose not. Then, for all $\eta < \theta^+$, the set
$x_\eta := \{\xi < \theta^{++} \mid g^{-1}[A_{\xi,\eta}] \cap \alpha \text{ is stationary in } \alpha\}$ has size $\leq \theta^+$.
So $X := \bigcup_{\eta<\theta^+} x_\eta$ has size $\leq \theta^+$, and we may fix $\xi \in \theta^{++} \setminus X$.
It follows that for all $\eta < \theta^+$, $g^{-1}[A_{\xi,\eta}] \cap \alpha$ is nonstationary in α.
Consequently, $g^{-1}[\bigcup_{\eta<\theta^+} A_{\xi,\eta}] \cap \alpha$ is nonstationary in α.
However, $\bigcup_{\eta<\theta^+} A_{\xi,\eta}$ contains a tail of θ^{++}, contradicting the fact that $\langle g(\beta) \mid \beta \in S^2_\alpha\rangle$ is strictly increasing and cofinal in θ^{++}. □

Step 4: Club-guessing

By Shelah's club-guessing theorem, we now fix a sequence $\langle C_\iota \mid \iota \in E^{\theta^{++}}_\theta \rangle$ such that, for every club $C \subseteq \theta^{++}$, there exists $\iota \in E^{\theta^{++}}_\theta$ such that $C_\iota \subseteq C \cap \iota$ and $\mathrm{otp}(C_\iota) = \theta$.

By Claim 3, for every $\alpha \in T_1$, let us fix $\eta_\alpha < \theta^+$ and $x_\alpha \in [\theta^{++}]^{\theta^{++}}$ such that, for all $\xi \in x_\alpha$, $g^{-1}[A_{\xi,\eta_\alpha}] \cap \alpha$ is stationary in α.
Then, fix $\iota_\alpha \in E^{\theta^{++}}_\theta$ such that $C_{\iota_\alpha} \subseteq \mathrm{acc}^+(x_\alpha) \cap \iota_\alpha$ and $\mathrm{otp}(C_{\iota_\alpha}) = \theta$.
By Fodor's lemma, fix a stationary $T_2 \subseteq T_1$, $\eta < \theta^+$ and $\iota \in E^{\theta^{++}}_\theta$ such that, for all $\alpha \in T_2$, $\eta_\alpha = \eta$ and $\iota_\alpha = \iota$.
As the elements of $\langle A_{\xi,\eta} \mid \xi < \theta^{++} \rangle$ are pairwise disjoint, we may fix a function $h : E^{\lambda^+}_\mu \to \theta$ such that, for all $\beta < \lambda^+$:

$$(g(\beta) \in A_{\xi,\eta} \ \&\ \xi < \iota) \implies h(\delta) = \sup(\mathrm{otp}(C_\iota \cap \xi)).$$

Step 5: Verification

For each $i < \theta$, let $S_i := h^{-1}\{i\}$.
We claim that $\langle S_i \mid i < \theta \rangle$ witnesses $\Pi(E^{\lambda^+}_{\mu}, \theta, E^{\lambda^+}_{\theta^{++}})$. Furthermore:

Claim 4

$\bigcap_{i<\theta} \mathrm{Tr}(S_i) \cap E^{\lambda^+}_{\theta^{++}}$ covers the stationary set T_2.

Proof. Fix arbitrary $\alpha \in T_2$ and $i < \theta$. We shall find a stationary subset $S' \subseteq E^{\alpha}_{\mu}$ such that $h[S'] = \{i\}$.
As $i < \theta = \mathrm{otp}(C_\iota)$, let ξ' denote the unique element of C_ι such that $\mathrm{otp}(C_\iota \cap \xi') = i$. Then, put $\xi := \min(x_\alpha \setminus (\xi' + 1))$.
As $C_\iota \subseteq \mathrm{acc}^+(x_\alpha)$, we have that $[\xi', \xi) \cap C_\iota = \{\xi'\}$.
Consequently, $\mathrm{otp}(C_\iota \cap \xi) = \mathrm{otp}(C_\iota \cap (\xi' + 1)) = i + 1$.
As $\eta = \eta_\alpha$ and $\xi \in x_\alpha$, the set $S' := g^{-1}[A_{\xi,\eta}] \cap \alpha$ is a stationary subset of E^{α}_{μ}. Finally, for each $\beta \in S'$, we have $g(\beta) \in A_{\xi,\eta}$, meaning that $h(\beta) = \sup(\mathrm{otp}(C_\iota \cap \xi)) = \sup(i + 1) = i$, as sought.

qed

A finer result

We also have a finer result that apply for arbitrary stationary $S \subseteq \lambda^+$ (rather than $S = E^{\lambda^+}_{\mu}$).

Theorem

Suppose that $\theta < \lambda$ are infinite cardinals with $\theta \neq \mathrm{cf}(\lambda)$, and S, T are subsets of λ^+ with $\mathrm{Tr}(S) \cap T \cap E^{\lambda^+}_{\theta}$ stationary.
Then any of the following implies that $\Pi(S, \theta, T)$ holds:

1. *λ is regular;*
2. *λ is a singular cardinal admitting a good scale, and $2^{\theta} \leq \lambda$.*

Good scale

A scale $\vec{f}$ for λ such that club many $\alpha \in E^{\lambda^+}_{>\mathrm{cf}(\lambda)}$ are good for $\vec{f}$.