# The collapse of the Continuum

Miguel Angel Mota (ITAM)

Joint work with David Asperó

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**Introduction**

The method of side conditions, invented by Todorcevic, describes a style of forcing in which elementary substructures are included in the conditions of a forcing poset $P$ to ensure properness of $P$ and hence, the preservation of $\omega_1$.

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If $q \in P$ and $N \prec H(\theta)$ with $|N| = \aleph_0$, then

1. $q$ is said to be $(N, P)$-generic iff for every dense subset $D$ of $P$ belonging to $N$, $D \cap N$ is predense below $q$.
2. $q$ is said to be strongly $(N, P)$-generic iff for every dense subset $D$ of $P \cap N$, $D$ is predense below $q$.

**R1** By elementarity, if $D$ is a dense subset of $P$ and $D, P \in N$, then $D \cap N$ is a dense subset of $P \cap N$. So, if $P \in N$, then $2 \Rightarrow 1$.
**R2** If $q$ is strongly $(N, P)$-generic, then $q$ forces that $N \cap G$ is a V-generic filter on the ctble. set $N \cap P$. So, $q$ adds a Cohen real.

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A typical condition of a forcing $P$ equipped with side cond. is a pair $(x, A)$ where $x$ is an approximation to the desired generic object and $A$ is a finite set of ctble. elementary substructures such that if $N \in A$, then $(x, A)$ is $(N, P)$-generic.

Friedman and Mitchell independently took the first step in generalizing this method from adding generic objects of size $\omega_1$ to adding larger objects by defining forcing posets with finite conditions for adding a club subset of $\omega_2$. Neeman was the first to simplify the side conditions of F. and M. by presenting a general framework for forcing on $\omega_2$ with side conditions.

The forcing posets of F, M, and N for adding a club of $\omega_2$ with finite cond. all force that $2^\omega = \omega_2$. In fact, they can be factored in many ways so that the quotient forcing also has strongly generic cond. in the intermediate extensions.

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Friedman asked whether it is possible to add a club subset of $\omega_2$ with finite conditions while preserving CH.

Together with John Krueger I solved this problem by defining a forcing poset which adds a club to a fat stationary set and falls in the class of the so called coherent adequate type forcings (Krueger and Mota, JML, 15, 2015).

Recall that a stationary set $S \subseteq \omega_2$ is said to be *fat* iff for every club $C \subseteq \omega_2$, $S \cap C$ contains a closed subset with o. t. $\omega_1 + 1$.

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We proved that our forcing preserves CH.

Moreover, we proved that any coherent adequate forcing on $H(\lambda)$ (meaning that our side conditions are ctble. elementary substructures of $H(\lambda)$) , where $2^\omega < \lambda$ is a cardinal of uncountable cofinality, collapses $2^\omega$ to have size $\omega_1$, preserves $(2^\omega)^+$, and forces CH.

Another common feature is that all these posets incorporate systems of countable structures with symmetry requirements as side conditions.

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## Definition

Let $T \subseteq H(\theta)$ and let $\mathcal{N}$ be a finite set of countable subsets of $H(\theta)$. We will say that $\mathcal{N}$ *is a T–symmetric system* iff

(*A*) For every $N \in \mathcal{N}$, $(N, \in, T) \prec (H(\theta), \in, T)$.

(*B*) Given distinct $N$, $N'$ in $\mathcal{N}$, if $\delta_N = \delta_{N'}$, then there is a unique isomorphism

$$\Psi_{N,N'} : (N, \in, T) \longrightarrow (N', \in, T)$$

Furthermore, $\Psi_{N,N'}$ is the identity on $N \cap N'$.

(*C*) $\mathcal{N}$ is closed under isomorphisms. That is, for all $N$, $N'$, $M$ in $\mathcal{N}$, if $M \in N$ and $\delta_N = \delta_{N'}$, then $\Psi_{N,N'}(M) \in \mathcal{N}$.

(*D*) For all $N$, $M$ in $\mathcal{N}$, if $\delta_M < \delta_N$, then there is some $N' \in \mathcal{N}$ such that $\delta_{N'} = \delta_N$ and $M \in N'$.

**Remark.** In all practical cases $\bigcup T = H(\theta)$, so $T$ does determine $H(\theta)$ in these cases.

The following lemmas are proved in TAMS, vol. 367, 2015 (Asperó and Mota).

### Lemma

*Let $T \subseteq H(\theta)$ and let $N$ and $N'$ be countable elementary substructures of $(H(\theta), \in, T)$. Suppose $\mathcal{N} \in N$ is a $T$–symmetric system and $\Psi : (N, \in, T) \longrightarrow (N', \in, T)$ is an isomorphism. Then $\Psi(\mathcal{N}) = \Psi``\mathcal{N}$ is also a $T$–symmetric system.*

## Lemma

*Let $T \subseteq H(\theta)$, let $\mathcal{N}$ be a $T$–symmetric system and let $N \in \mathcal{N}$. Then the following holds.*

1. *$\mathcal{N} \cap N$ is a $T$–symmetric system.*
2. *Suppose $\mathcal{N}^* \in N$ is a $T$–symmetric system such that $\mathcal{N} \cap N \subseteq \mathcal{N}^*$. Let*

$$\mathcal{M} = \mathcal{N} \cup \bigcup \{\Psi_{N,N'} \text{“} \mathcal{N}^* \,:\, N' \in \mathcal{N},\, \delta_{N'} = \delta_N\}$$

   *Then $\mathcal{M}$ is the $\subseteq$–minimal $T$–symmetric system $\mathcal{W}$ such that $\mathcal{N} \cup \mathcal{N}^* \subseteq \mathcal{W}$.*

Given $T \subseteq H(\theta)$ and $T$–symmetric systems $\mathcal{N}_0$, $\mathcal{N}_1$, let us write $\mathcal{N}_0 \cong \mathcal{N}_1$ iff

- $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = R$ and
- for some $m < \omega$, there are enumerations $(N_i^0)_{i<m}$ and $(N_i^1)_{i<m}$ of $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively, together with an isomorphism between

  $$\langle \bigcup \mathcal{N}_0, \in, T, R, N_i^0 \rangle_{i<m}$$

  and

  $$\langle \bigcup \mathcal{N}_1, \in, T, R, N_i^1 \rangle_{i<m}$$

  which is the identity on $R$.

## Lemma

*Let $T \subseteq H(\theta)$ and let $\mathcal{N}_0$ and $\mathcal{N}_1$ be $T$–symmetric systems. Suppose $\mathcal{N}_0 \cong \mathcal{N}_1$. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a $T$–symmetric system.*

### Definition

The poset $\mathcal{P}_0$ is the set of all the $T$–symmetric systems. Given $q_1$ and $q_0$ in $\mathcal{P}_0$, $q_1 \leq_{\mathcal{P}_0} q_0$ iff $q_0 \subseteq q_1$.

### Corollary

1. *$\mathcal{P}_0$ is (strongly) proper.*
2. *(CH) If there is a bijection between $\theta$ and $H(\theta)$ which is definable in $(H(\theta), \in, T)$, then $\mathcal{P}_0$ is $\aleph_2$–Knaster.*
3. *(CH) If there is a bijection between $\theta$ and $H(\theta)$ which is definable in $(H(\theta), \in, T)$, then $\mathcal{P}_0$ preserves CH (Asperó and Mota, APAL, 2016)*

**Proof of (1).** Suppose that $\kappa$ is regular and $N^*$ is a ctble. elem. substr. of $H(\kappa)$ s. t. $\mathcal{P}_0$ and the cond. $s$ are in $N^*$. Then, letting $N = N^* \cap H(\theta)$ and $s' = s \cup \{N\}$, $s'$ is $(N^*, \mathcal{P}_0)$–generic.

Let $E$ be a dense subset of $\mathcal{P}_0$ in $N^*$. It suffices to show that there is some condition in $E \cap N^*$ compatible with $s'$. Notice first that $s' \cap N \in \mathcal{P}_0$. Hence, we may find a condition $s^\circ \in E \cap N$ extending $s' \cap N$. Now let

$$s^* = s' \cup \{\Psi_{N,\overline{N}}(M) \,:\, M \in q^\circ, \overline{N} \in s', \delta_{\overline{N}} = \delta_N\}.$$

So, $s^*$ is a condition in $\mathcal{P}_0$ extending both $s'$ and $s^\circ$.

**Proof of (2).** Suppose that $s_\xi = \{N_i^\xi : i < m\}$ is a $\mathcal{P}_0$–condition for each $\xi < \omega_2$. By CH we may assume that $\{\bigcup_{i<m} N_i^\xi : \xi < \omega_2\}$ forms a $\Delta$–system with root $X$. Moreover, by CH we may assume, for all $\xi, \xi' < \omega_2$, that the structures $\langle \bigcup_{i<m} N_i^\xi, \in, P, X, N_i^\xi \rangle_{i<m}$ and $\langle \bigcup_{i<m} N_i^{\xi'}, \in, P, X, N_i^{\xi'} \rangle_{i<m}$ are isomorphic and that the isomorphism fixes $X$.

The first assertion follows from the fact that there are only $\aleph_1$–many iso. types for such structures. For the second assertion note that, if $\Psi$ is the unique isomorphism between $\langle \bigcup_{i<m} N_i^{\xi}, \in, T, X, N_i^{\xi}\rangle_{i<m}$ and $\langle \bigcup_{i<m} N_i^{\xi'}, \in, T, X, N_i^{\xi'}\rangle_{i<m}$, then the restriction of $\Psi$ to $X \cap \theta$ has to be the identity on $X \cap \theta$. Since there is a bijection between $\theta$ and $H(\theta)$ which is definable in $(H(\theta), \in, T)$, we have that $\Psi$ fixes $X$ if and only if it fixes $X \cap \theta$. It follows that $\Psi$ fixes $X$. Hence, for all $\xi, \xi' < \omega_2$, $s_\xi \cup s_{\xi'}$ extends both $s_\xi$ and $s_{\xi'}$.

**Proof of (3).** Suppose $s \in \mathcal{P}_0$, $\dot{r}_\alpha$ (for $\alpha < \omega_2$) are $\mathcal{P}_0$–names , and $s$ forces that $\dot{r}_\alpha$, for $\alpha < \omega_2$, are pairwise distinct reals . By the $\aleph_2$–chain condition of $\mathcal{P}_0$ we may assume that each $\dot{r}_\alpha$ is in $H(\theta)$. Let $\kappa$ be a regular cardinal such that $\mathcal{P}_0 \in H(\kappa)$. For each $\alpha$ let $N_\alpha$ be such that $\{q, \dot{r}_\alpha\} \in N_\alpha$ and $N_\alpha$ is a countable elementary substructure of $(H(\theta), \in, P, \mathcal{S}_P)$. We can also assume that for each $\alpha$, there is a countable elementary substructure $N^*_\alpha \prec H(\kappa)$ such that $N_\alpha = H(\theta) \cap N^*_\alpha$. By CH, there are distinct $\alpha$, $\alpha'$ such that $(N_\alpha, \in, P, \mathcal{P}_0, s, \dot{r}_\alpha)$ and $(N_\alpha, \in, P, \mathcal{P}_0, s, \dot{r}_\alpha)$ are isomorphic.

By the above lemmas we may also assume that $s \cup \{N_\alpha, N_{\alpha'}\}$ is a $\mathcal{P}_0$–condition. So, $s \cup \{N_\alpha, N_{\alpha'}\}$ is $(N^*_\alpha, \mathcal{P}_0)$–generic and $(N^*_{\alpha'}, \mathcal{P}_0)$–generic. Let $\Psi$ be the unique isomorphism between $N_\alpha$ and $N_{\alpha'}$ and note that for every natural number $n$ and for every condition $s'$ $\mathcal{P}_0$–extending $s \cup \{N_\alpha, N_{\alpha'}\}$, there are conditions $s''$ and $r$ such that $r \in N_\alpha$, $r$ decides the $n$th value of $\dot{r}_\alpha$ and $s''$ is a common $\mathcal{P}_0$–extension of $r$ and $s'$. Since symmetric systems are closed under isomorphism, $s''$ also $\mathcal{P}_0$–extends $\Psi(r) \in N_{\alpha'}$. By correctness of $\Psi$, $\Psi(r)$ forces that the $n$th value of $\Psi(\dot{r}_\alpha) = \dot{r}_{\alpha'}$ is equal to the $n$th value of $\dot{r}_\alpha$. So, $s \cup \{N_\alpha, N_{\alpha'}\}$ forces that $\dot{r}_\alpha = \dot{r}_{\alpha'}$. This is a contradiction.

## In the absence of convenient cardinal assumptions

Of course, if $T$ codes a well-ordering of $H(\theta)$ and $2^\omega < \theta$, then any two isomorphic $N_1, N_2 \prec (H(\theta), \in, T)$ have the same reals which implies that $\mathcal{P}_0$ collapses $(2^\omega)^V$ to $\omega_1$. Moreover,

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## Applications in the context of iterated forcing

Something one may naturally envision at this point is the possibility to build a suitable forcing iteration with systems of models as side conditions while strengthening the symmetry constraints, so as to make them apply not only to the side condition part of the forcing but also to the working parts; one would hope to exploit the above idea in order to show that the iteration thus constructed preserves *CH*, and would of course like to be able to do that while at the same time forcing some interesting statement.

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### Definition

*Measuring* holds if and only if for every sequence $\vec{C} = (C_\delta : \delta \in \omega_1)$, if each $C_\delta$ is a closed subset of $\delta$ in the order topology, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq C_\delta$, or
- $(C \setminus \alpha) \cap C_\delta = \emptyset$.

That is, a tail of $(C \cap \delta)$ is either contained in or disjoint from $C_\delta$.

This principle is of course equivalent to its restriction to club-sequences $\vec{C}$ on $\omega_1$.

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