Critical Cardinals Redux

Joint work with Yair Hayut

Asaf Karagila

University of East Anglia

22 January 2019

Arctic Set Theory 4, 2019

Definition

We say that a cardinal κ is a *critical cardinal* if it is the critical point of an elementary embedding $j\colon V_{\kappa+1} \to M$, where M is a transitive set.

Assuming ZFC

- we can replace $V_{\kappa+1}$ by V,
- or we can assert the existence of a κ-complete free ultrafilter on κ,
- or we can require said ultrafilter to be normal,
- or we can require that M is an ultrapower,
- or we can require that M is κ^+-closed.

Question

Suppose that κ is a critical cardinal and $j\colon V_{\kappa+1} \to M$ is an elementary embedding with critical point κ.

1. *Can we assume that M is ${\leqslant}V_\kappa$-closed?*
2. *Can we assume that M is κ^+-closed?*
3. *Can we assume that M is σ-closed?*
4. *Can we assume that M is an ultrapower?*
5. *Can we assume that j is an embedding from V?*

There are many natural questions, and it is unclear whether or not which can be answered positively or how difficult counterexamples might be.

We can still prove some things.

Proposition

If κ is a critical cardinal, then the following holds:

1. *κ is a regular cardinal,*
2. *κ is a strong limit cardinal,*
3. *V_κ is a model of ZF_2,*
4. *κ carries a normal measure derived from the embedding,*
5. *this normal measure concentrates on weakly critical cardinals.*

Without choice, of course, ω_1 can be measurable. Or a measurable cardinal might not have a normal measure. Or a measurable cardinal might just not be the critical point of an embedding.

Wait. *Weakly* critical cardinals?

Definition

We say that κ is a *weakly critical cardinal* if for every $A \subseteq V_\kappa$ there is a transitive set X such that $\kappa, V_\kappa, A \in X$ and an elementary embedding $j\colon X \to M$ to a transitive set M, such that κ is the critical point of j.

Assuming ZFC, this is equivalent to weak compactness, but as usual, without choice the situation is more complex.

Question (Kaplan)

Can the first measurable cardinal be the first weakly critical cardinal?

(Probably.)

Proposition

κ is a weakly critical cardinal if and only if for every $A \subseteq V_\kappa$, there is a transitive, elementary end extension of $\langle V_\kappa, \in, A\rangle$.

Sketch of Proof.

Assuming κ is weakly critical, and $A \subseteq V_\kappa$ is some set, and X is a transitive set witnessing that κ is weakly critical. Let $j\colon X \to M$ be the elementary embedding. Then it is not hard to check that $\langle j(V_\kappa), \in, j(A)\rangle$ is indeed the wanted end extension.

In the other direction, if $A \subseteq V_\kappa$, let $\langle W, \in, B\rangle$ be an elementary end extension of V_κ with A as a predicate. Take $X = V_\kappa \cup \{\kappa, V_\kappa, A\}$ and $M = W \cup \{W \cap \mathrm{Ord}, W, B\}$ and consider $j\colon X \to M$ defined in the obvious way. $\square$

Using this equivalence it is very easy to prove that if κ is weakly critical, then:

1. κ is regular,
2. κ is a strong limit cardinal,
3. κ is Mahlo, in fact κ^+-Mahlo,
4. κ has the tree property, etc.

This is in contrast to ω_1, for example, having the tree property or other characterizations of weak compactness.

It is also very clear that a critical cardinal is weakly critical. And as mentioned, has a measure 1 subset of critical cardinals below it.

Going back to critical cardinals...

Theorem (Hayut–K.)

If κ is a supercompact cardinal, then there is a symmetric extension such that κ remains a critical cardinal, but $\mathrm{cf}(\kappa^+) = \omega$.

The argument goes by supercompact Radin forcing which collapses $\kappa^{+\omega}$ and preserves the measurability of κ.

Of course replace ω by any regular cardinal $\mu < \kappa$.

Question

1. *Can we replace ω by κ itself?*
2. *Can we make κ^+ regular, but preserve combinatorial properties of κ^+?*
3. *Specifically, can κ^+ be measurable?*

If $\mathbb{P}$ is a forcing, and π is an automorphism of $\mathbb{P}$, then π extends to $\mathbb{P}$-names via this recursive definition:

$$\pi\dot{x} = \{\langle \pi p, \pi\dot{y}\rangle \mid \langle p, \dot{y}\rangle \in \dot{x}\}.$$

Let $\mathscr{G}$ be a subgroup of $\operatorname{Aut}(\mathbb{P})$, and let $\mathscr{F}$ be a normal filter of subgroups of $\mathscr{G}$.

- $\dot{x}$ is **symmetric** if $\{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\} \in \mathscr{F}$.
- HS denotes the class of **hereditarily symmetric** names.
- If G is a V-generic filter for $\mathbb{P}$, then $\mathsf{HS}^G = \{\dot{x}^G \mid \dot{x} \in \mathsf{HS}\}$ is called a **symmetric extension** of V. It is a transitive subclass of $V[G]$ which contains V and satisfies ZF.
- We say that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ is a **symmetric system** if $\mathscr{G}$ is an automorphism group of $\mathbb{P}$ and $\mathscr{F}$ is a normal filter of subgroups of $\mathscr{G}$.

Theorem (K.; Usuba)

If M is a symmetric extension of V, then V is definable in M, and the statement "The universe is a symmetric extension of V using the symmetric system $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$" is a first-order statement.

Let us fix $j\colon V \to M$ with κ its critical point.

Theorem (Silver's criterion)

j can be lifted to $j\colon V[G] \to M[H]$ if and only if $j(G) = H$.

Suppose now that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is a symmetric system and W is the symmetric extension it defines. If N is the corresponding symmetric extension of M by $j(\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle)$, we want to lift j to an embedding from W to N.

And we want this extension to be sufficiently amenable to W to witness that κ is still critical. In particular that means that $N \subseteq W$, at least to a sufficiently high rank.

In the settings of Silver's theorem, in order for the lifting to be amenable to $V[G]$, we need that $H \in V[G]$. But G is (usually) not an element of W, so we cannot fully expect $H \in W$ either in the case of symmetric extensions.

Definition

Let $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ be a symmetric system.

1. We say that $D \subseteq \mathbb{P}$ is a *symmetric subset* if there is some $H \in \mathscr{F}$ such that for all $\pi \in H$, $\pi " D = D$.
2. We say that G is *symmetrically V-generic* if for every symmetrically dense open $D \in V$, $D \cap G \neq \varnothing$.

This notion of genericity is "the right notion" when it comes to symmetric extensions. In particular, if there is a symmetrically M-generic filter in W for $j(\mathbb{P})$, then the embedding can be lifted in an amenable way.

Almost. We need the embedding to also play nice with the rest of the symmetric system, not just with the generic filters.

But we start by requiring that $j(\mathbb{P}) \cong \mathbb{P} * \dot{\mathbb{Q}}$ and there is some $H \in W$ which is symmetrically M-generic for $\mathbb{Q}$.

Proposition

Suppose that $\left\{\left\langle\dot{x}, j(\dot{x})^H\right\rangle^\bullet \mid \dot{x} \in \mathsf{HS}\right\}^\bullet$ *is a symmetric class. Then j can be lifted to the symmetric extensions.*

Here $j(\dot{x})^H$ denotes a $\mathbb{P}$-name partial interpretation of $j(\dot{x})$ by the name for H.

But for the conditions of the proposition to hold we need to verify that $j(\dot{x})^H \in \mathsf{HS}$, and that if $\pi \in \mathscr{G}$, then $\pi(\dot{a}^H) = (j(\pi)\dot{a})^{\pi H}$. Indeed, if this is the case, and $\pi\dot{H} = \dot{H}$ for all π, then

$$\pi(\left\langle\dot{x}, j(\dot{x})^H\right\rangle^\bullet) = \left\langle\pi\dot{x}, \pi(j(\dot{x})^H)\right\rangle^\bullet = \left\langle\pi\dot{x}, (j(\pi)j(\dot{x}))^{\pi H}\right\rangle^\bullet = \left\langle\pi\dot{x}, j(\pi\dot{x})^H\right\rangle^\bullet.$$

So we need to find a condition that ensures that $\pi(\dot{x}^H) = (j(\pi)\dot{x})^{\pi H}$. And we have one. Which is awful. But can summarily be stated inaccurately as follows.

Inaccurately Stated Theorem (Hayut–K.)

Suppose that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is a symmetric system and $j \colon V \to M$ is an elementary embedding. Then j lifts to the symmetric extension, W if:

1. $j(\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle) \cong \langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle * \langle \dot{\mathbb{Q}}, \dot{\mathscr{H}}, \dot{\mathscr{K}} \rangle^{\bullet}$,
2. *there is* $\dot{H} \in \mathsf{HS}$ *which is a name for a symmetrically M-generic filter for* $\mathbb{Q}$,
3. *the name for* $\dot{H}$ *is essentially stable under* $\mathscr{H}$,
4. *and* $j``\mathscr{F}$ *is a basis for* $j(\mathscr{F})$.

The last requirement is the most strenuous of the four. But it is also reasonable.

When $\mathscr{F}$ is κ-complete, then $j(\mathscr{F})$ is $j(\kappa)$-complete, which lets us intersect $j``\mathscr{F}$ into a single group in $j(\mathscr{F})$. This means that essentially the copy of $\mathbb{P}$ in $j(\mathbb{P})$ is fixed pointwise, so G must be present somehow in W.

Going back to the question about a regular successor of a critical cardinal, this would normally be arranged by having $\mathscr{F}$ behave in a sufficiently complete form.

We can also find examples where j lifts between the generic extensions, but not in a way that is amenable to the symmetric extensions.

Suppose that κ a measurable cardinal immune to adding a Cohen subset.

Consider the symmetric system which adds κ Cohen subsets to κ, $\mathscr{G}$ permutes their indices, and $\mathscr{F}$ is given by fixing $<\kappa$ indices. Let W be the symmetric extension.

Then in W the following hold:

1. $W_\kappa = V_\kappa$,
2. and $W_{\kappa+1}$ cannot be well-ordered.

If j lifts, then $j(W_\kappa)$ satisfies AC and in particular must know of a well-ordering of $W_{\kappa+1}$. But this is impossible *inside* of W, so the lifting is not amenable.

(Note that $\mathscr{F}$ here is κ-complete, by the way.)

So what are the goals we hope to achieve?

Goal

Improve the lifting theorem. Both by remove requirement (4) as much as possible, and ideally by finding an exact equivalent condition for when an embedding can be lifted amenably.

Doing so will allow us to obtain many new results about successors of critical points. In addition, if κ is critical and κ^+ is measurable, then it is a step towards finding a symmetric extension in which ω_2 and ω_3 are measurable.

Goal

Learn how to control the closure of the lifted embeddings, and learn how to lift many (unrelated) embeddings at once.

Doing this will allow us to get one step closer into measuring the consistency strength of nontrivial-supercompact cardinals, and could help to clarify the fog around the HOD Conjecture.

Recall Woodin's theorem: If δ is a supercompact, then there is a forcing which collapses δ to be ω_1 and DC holds. We say that δ is a *nontrivial-supercompact* if Woodin's forcing *does not* force AC.

Can we start with $\mathsf{ZFC} + \mathsf{LC}$ and construct a model with a nontrivial-supercopmact using a symmetric extension?

Goal

Iterations.(Of symmetric extensions.)

Symmetric extensions can be iterated, but it is unclear if an iteration of symmetric extensions is itself a symmetric extension. Extending the lifting criterion to iterations of symmetric extensions can be instrumental in the study of choiceless large cardinals, since iterations sometimes help simplify constructions that are otherwise very complicated.

Thank you!