More ZFC inequalities between cardinal invariants

Vera Fischer

University of Vienna

January 2019

Outline: Higher Analogues

1. Eventual difference and $\mathfrak{a}_e(\kappa)$, $\mathfrak{a}_p(\kappa)$, $\mathfrak{a}_g(\kappa)$;
2. Generalized Unsplitting and Domination;

Eventual Difference

Almost disjointness

$\mathfrak{a}(\kappa)$ is the min size of a max almost disjoint $\mathscr{A} \subseteq [\kappa]^\kappa$ of size $\geq \kappa$

Relatives

- $\mathfrak{a}_e(\kappa)$ is the min size of max, eventually different family $\mathscr{F} \subseteq {}^\kappa\kappa$,
- $\mathfrak{a}_p(\kappa)$ is the min size of a max, eventually different family $\mathscr{F} \subseteq S(\kappa) := \{f \in {}^\kappa\kappa : f \text{ is bijective}\}$,
- $\mathfrak{a}_g(\kappa)$ is the min size of a max, almost disjoint subgroup of $S(\kappa)$.

What we still do not know...

Even though Con($\mathfrak{a} < \mathfrak{a}_g$), both

- the consistency of $\mathfrak{a}_g < \mathfrak{a}$, as well as
- the inequality $\mathfrak{a} \leq \mathfrak{a}_g$ (in ZFC)

are open.

Roitman Problem

Is it consistent that $\mathfrak{d} < \mathfrak{a}$?

- Yes, if $\aleph_1 < \mathfrak{d}$ (Shelah's template construction).
- Open, if $\aleph_1 = \mathfrak{d}$.

Is it consistent that $\mathfrak{d} = \aleph_1 < \mathfrak{a}_g$?

One of the major differences between $\mathfrak{a}$ and its relatives, is their relation to $\mathfrak{non}(\mathcal{M})$.

- While $\mathfrak{a}$ and $\mathfrak{non}(\mathcal{M})$ are independent,
- $\mathfrak{non}(\mathcal{M}) \leq \mathfrak{a}_e, \mathfrak{a}_p, \mathfrak{a}_g$ (Brendle, Spinas, Zhang),

Thus in particular, consistently $\mathfrak{d} = \aleph_1 < \mathfrak{a}_g = \aleph_2$.

Uniformity of the Meager Ideal: Higher Analogue

For κ regular uncountable, define $\mathfrak{nm}(\kappa)$ to be the least size of a family $\mathscr{F} \subseteq {}^\kappa\kappa$ such that $\forall g \in {}^\kappa\kappa \exists f \in \mathscr{F}$ with $|\{\alpha \in \kappa : f(\alpha) = g(\alpha)\}| = \kappa$.

Theorem (Hyttinen)

If κ is a successor, then $\mathfrak{mn}(\kappa) = \mathfrak{b}(\kappa)$.

Theorem (Blass, Hyttinen, Zhang)

Let κ be regular uncountable. Then $\mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa), \mathfrak{a}_e(\kappa), \mathfrak{a}_p(\kappa), \mathfrak{a}_g(\kappa)$;

Corollary

Thus for κ successors, $\mathfrak{nm}(\kappa) = \mathfrak{b}(\kappa) \leq \mathfrak{a}(\kappa), \mathfrak{a}_e(\kappa), \mathfrak{a}_p(\kappa), \mathfrak{a}_g(\kappa)$.

Roitman in the Uncountable

Theorem (Blass, Hyttinen and Zhang)

Let $\kappa \geq \aleph_1$ be regular uncountable. Then

$$\mathfrak{d}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}(\kappa) = \kappa^+.$$

Roitman in the Uncountable

The cofinitary groups analogue

- Clearly, the result does not have a cofinitary group analogue for $\kappa = \aleph_0$, since $\mathfrak{d} = \aleph_1 < \mathfrak{a}_g = \mathfrak{a}_g(\aleph_0) = \aleph_2$ is consistent.
- Nevertheless the question remains of interest for uncountable κ: Is it consistent that

$$\mathfrak{d}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}_g(\kappa) = \kappa^+?$$

Club unboundedness

Theorem (Raghavan, Shelah, 2018)

Let κ be regular uncountable. Then $\mathfrak{b}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}(\kappa) = \kappa^+$.

Club unboundedness

1. Let κ be regular uncountable. For $f, g \in {}^\kappa\kappa$ we say that $f \leq_{cl} g$ iff $\{\alpha < \kappa : g(\alpha) < f(\alpha)\}$ is non-stationary.
2. $\mathscr{F} \subseteq {}^\kappa\kappa$ is $\leq_{cl}$-unbounded if $\neg\exists g \in {}^\kappa\kappa \forall f \in \mathscr{F}(f \leq_{cl} g)$.
3. $\mathfrak{b}_{cl}(\kappa) = \min\{|F| : F \subseteq {}^\kappa\kappa$ and F is cl-unbounded$\}$

Theorem (Cummings, Shelah)

If κ is regular uncountable then $\mathfrak{b}(\kappa) = \mathfrak{b}_{\text{cl}}(\kappa)$.

Higher eventually different analogues

Theorem(F., D. Soukup, 2018)

Suppose $\kappa = \lambda^+$ for some infinite λ and $\mathfrak{b}(\kappa) = \kappa^+$. Then $\mathfrak{a}_e(\kappa) = \mathfrak{a}_p(\kappa) = \kappa^+$. If in addition $2^{<\lambda} = \lambda$, then $\mathfrak{a}_g(\kappa) = \kappa^+$.

Remark

The case of $\mathfrak{a}_e(\kappa)$ has been considered earlier. The above is a strengthening of each of the following:

- $\mathfrak{d}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}_e(\kappa) = \kappa^+$ for κ successor (Blass, Hyttinen, Zhang)
- $\mathfrak{b}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}_e(\kappa) = \kappa^+$ proved by Hyttinen under additional assumptions.

Outline: $\mathfrak{b}(\kappa) = \kappa^+ \Rightarrow \mathfrak{a}_e(\kappa) = \kappa^+$

- For each $\lambda : \lambda \leq \alpha < \lambda^+ = \kappa$ fix a bijection

$$d_\alpha : \alpha \to \lambda.$$

- For each $\delta : \lambda^+ = \kappa \leq \delta < \kappa^+$ fix a bijection

$$e_\delta : \kappa \to \delta.$$

- Let $\{f_\delta : \delta < \kappa^+\}$ witness $\mathfrak{b}_{\mathrm{cl}}(\kappa) = \kappa^+$.

- We will define functions $h_{\delta,\zeta} \in {}^{\kappa}\kappa$ for $\delta < \kappa^+$, $\zeta < \lambda$ by induction on δ, simultaneously for all $\zeta < \lambda$.
- Thus, suppose we have defined $h_{\delta',\zeta}$ for $\delta' < \delta$, $\zeta < \lambda$.
- Let $\mu < \kappa$. We want to define $h_{\delta,\zeta}(\mu)$ for each $\zeta \in \lambda$. Let

$$\mathbb{H}_\delta(\mu) = \{h_{\delta',\zeta'} : \delta' \in \text{ran}(e_\delta \restriction \mu), \zeta' \in \lambda\}.$$

Then, since $e_\delta : \kappa \to \delta$ is a bijection, $|\text{ran}(e_\delta \restriction \mu)| \leq \lambda$ and so $\mathbb{H}_\delta(\mu)$, as well as

$$H_\delta(\mu) = \{h(\mu) : h \in \mathbb{H}_\delta(\mu)\}$$

are of size $< \kappa$.

- Then define:

$$f^*_\delta(\mu) = \max\{f_\delta(\mu), \min\{\alpha \in \kappa : |\alpha \backslash H_\delta(\mu)| = \lambda\}\} < \kappa.$$

- Now, $|f^*_\delta(\mu) \backslash H_\delta(\mu)| = \lambda$ and so, we have enough space to define the values $h_{\delta,\zeta}(\mu)$ distinct for all $\zeta < \lambda$.
- More precisely, for each $\zeta < \lambda$, define $h_{\delta,\zeta}(\mu) := \beta$ where β is such that

$$d_{f^*_\delta(\mu)}[\beta \cap (f^*_\delta(\mu) \backslash H_\delta(\mu))]$$

  is of order type ζ. We say that $h_{\delta,\zeta}(\mu)$ is the ζ-th element of $f^*_\delta(\mu) \backslash H_\delta(\mu)$ with respect to $d_{f^*_\delta(\mu)}$.

Claim: $\{h_{\delta,\zeta}\}_{\delta<\kappa^+,\zeta<\lambda}$ is κ-e.d.

- Case 1: Fix $\delta < \kappa^+$. If $\zeta \neq \zeta'$, then by definition $h_{\delta,\zeta}(\mu) \neq h_{\delta,\zeta'}(\mu)$ for each $\mu < \kappa$.
- Case 2: Let $\delta' < \delta < \kappa^+$ and $\zeta, \zeta' < \lambda$ be given. Since $e_\delta : \kappa \to \delta$ is a bijection, there is $\mu_0 \in \kappa$ such that $\delta' \in \text{range}(e_\delta \restriction \mu_0)$.
- But then for each $\mu \geq \mu_0$, $h_{\delta',\zeta'} \in \mathbb{H}_\delta(\mu)$ and so $h_{\delta',\zeta'}(\mu) \in H_\delta(\mu)$.
- However $h_{\delta,\zeta} \in \kappa \backslash H_\delta(\mu)$ and so $h_{\delta',\zeta'}(\mu) \neq h_{\delta,\zeta}(\mu)$.

Claim: $\{h_{\delta,\zeta}\}_{\delta<\kappa^+,\zeta<\lambda}$ is κ-med.

Let $h \in {}^\kappa\kappa$ and $\delta < \kappa^+$ such that

$$S = \{\mu \in \kappa : h(\mu) < f_\delta(\mu)\}$$

is stationary. There is stationary $S_0 \subseteq S$ such that

1. $h(\mu) \in H_\delta(\mu)$ for all $\mu \in S_0$, or
2. $h(\mu) \notin H_\delta(\mu)$ for all $\mu \in S_0$.

We will see that in either case, there are δ, ζ such that

$$h_{\delta,\zeta}(\mu) = h(\mu)$$

for stationarily many $\mu \in S_0$.

Case 1: $h(\mu) \in H_\delta(\mu)$ for all $\mu \in S_0$

Recall:

- $\mathbb{H}_\delta(\mu) = \{h_{\delta',\zeta'} : \delta' \in \mathrm{ran}(e_\delta \restriction \mu), \zeta' \in \lambda\}$, and
- $H_\delta(\mu) = \{h(\mu) : h \in \mathbb{H}_\delta(\mu)\}$.

Now:

- For each $\mu \in S_0$ there are $\eta_\mu < \mu$, $\zeta_\mu < \lambda$ such that $h(\mu) = h_{e_\delta(\eta_\mu),\zeta_\mu}(\mu)$.
- By Fodor's Lemma we can find a stationary $S_1 \subseteq S_0$ such that for all $\mu \in S_1$, $\eta_\mu = \eta$ for some $\eta < \mu$.
- Then for $\delta' = e_\delta(\eta)$ we can find stationarily many $\mu \in S_1$ such that $\zeta_\mu = \zeta'$ for some ζ', and so
- for stationarily many μ in S_1 we have $h(\mu) = h_{\delta',\zeta'}(\mu)$.

Case 2: $h(\mu) \notin H_\delta(\mu)$ for all $\mu \in S_0$

Recall:

$f^*_\delta(\mu) = \max\{f_\delta(\mu), \min\{\alpha \in \kappa : |\alpha \backslash H_\delta(\mu)| = \lambda\}\} < \kappa$

Now:

- For each $\mu \in S_0$, since $h(\mu) < f_\delta(\mu)$ and $f_\delta(\mu) \leq f^*_\delta(\mu)$, we have $h(\mu) \in f^*_\delta(\mu) \backslash H_\delta(\mu)$.
- Thus, for each $\mu \in S_0 \backslash (\lambda + 1)$ there is $\zeta_\mu < \lambda \leq \mu$ such that $h(\mu)$ is the ζ_μ-th element of $f^*_\delta(\mu) \backslash H_\delta(\mu)$ with respect to $d_{f^*_\delta(\mu)}$.
- By Fodor's Lemma, there is a stationary $S_1 \subseteq S_0$ such that for each $\mu \in S_1$, $\zeta = \zeta_\mu$ for some ζ and so for all $\mu \in S_1$ we have $h(\mu) = h_{\delta,\zeta}(\mu)$. □

Question

- Is it true that $\mathfrak{b}(\kappa) = \kappa^+$ implies that $\mathfrak{a}_e(\kappa) = \mathfrak{a}_p(\kappa) = \kappa^+$ if κ is not a successor?

Definition

Let κ be regular uncountable.

- A family $F \subseteq [\kappa]^\kappa$ is splitting if for every $B \in [\kappa]^\kappa$ there is $A \in F$ such that $|B \cap A| = |B \cap (\kappa \backslash A)| = \kappa$, i.e. A splits B. Then

$$\mathfrak{s}(\kappa) := \min\{|F| : F \text{ is splitting}\}.$$

- A family $F \subseteq [\kappa]^\kappa$ is unsplit if there is no $B \in [\kappa]^\kappa$ which splits every element of F. Then

$$\mathfrak{r}(\kappa) := \min\{|F| : F \text{ is unsplit }\}.$$

Theorem (Raghavan, Shelah)

Let κ be regular uncountable. Then $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

- Thus splitting and unboundedness at κ behave very differently than splitting and unboundedness at ω, as it is well known that $\mathfrak{s}$ and $\mathfrak{b}$ are independent.
- Is it true that for every regular uncountable κ, $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$?

Theorem (Raghavan, Shelah)

Let $\kappa \geq \beth_\omega$ be regular. Then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.

Club domination

1. $\mathscr{F} \subseteq {}^\kappa\kappa$ is $\leq_{cl}$-dominating if $\forall g \in {}^\kappa\kappa \exists f \in \mathscr{F}(g \leq_{cl} f)$.
2. $\mathfrak{d}_{\text{cl}}(\kappa) = \min\{|F| : F \subseteq {}^\kappa\kappa \wedge F \text{is cl-dominating}\}$.

Almost always the same

Theorem (Cummings, Shelah)

$\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$ whenever $\kappa \geq \beth_\omega$ regular.

The RS-property

Notation

For κ be regular uncountable and $A \in [\kappa]^\kappa$, let $s_A : \kappa \to A$ be defined as follows:

$$s_A(\alpha) = \min(A \setminus (\alpha + 1)).$$

Definition

We say that $F \subseteq [\kappa]^\kappa$ has the RS-property if for every club $E_1 \subseteq \kappa$, there is a club $E_2 \subseteq E_1$ such that for every $A \in F$,

$$A \not\subseteq^* \bigcup_{\xi \in E_2} [\xi, s_{E_1}(\xi)).$$

Outline: $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ for $\kappa \geq \beth_\omega$ regular

Take $F \subseteq [\kappa]^\kappa$ unsplit of size $\mathfrak{r}(\kappa)$. With F we will associate a dominating family of size $\leq \mathfrak{r}(\kappa)$.

Suppose F does not have the RS-property.

- Thus $\exists$ club E_1 such that $\forall$ club $E_2 \subseteq E_1$ there is $A \in F$ with $A \subseteq^* \bigcup_{\xi \in E_2} [\xi, s_{E_1}(\xi))$.
- We will show that $\{s_A \circ s_{E_1} : A \in F\}$ is $\leq^*$-dominating.
- Let $f \in {}^\kappa\kappa$ be arbitrary. Take $E_f = \{\xi \in E_1 : \xi \text{ is closed under } f\}$.
- Then E_f is a club and since F does not have the RS-property, there is $A \in F$ and $\delta \in \kappa$ such that $A \backslash \delta \subseteq \bigcup_{\xi \in E_f} [\xi, s_{E_1}(\xi))$.
- Then $\forall \zeta \geq \delta$, $f(\zeta) < (s_A \circ s_{E_1})(\zeta)$.
- Since f was arbitrary, $\{s_A \circ s_{E_1} : A \in F\}$ is indeed $\leq^*$-dominating.

Suppose F does have the RS-property.

- That is, for every club E_1, there is a club $E_2 \subseteq E_1$ such that for all $A \in F$, $A \not\subseteq^* \bigcup_{\xi \in E_2} [\xi, s_{E_1}(\xi))$.
- We will show that $\{s_A : A \in F\}$ is $\leq_{cl}$-dominating.
- Take $f \in {}^\kappa\kappa$ and let E_f be an f-closed club. Pick E_2-given by RS.
- If for all $A \in F$, $|A \cap \bigcup_{\xi \in E_2} [\xi, s_{E_f}(\xi))| = \kappa$, then $\bigcup_{\xi \in E_2} [\xi, s_{E_f}(\xi))$ splits F, contradicting F is unsplit.

- Thus there are $A \in F$, $\delta < \kappa$ with $A \backslash \delta \subseteq \kappa \backslash \bigcup_{\xi \in E_2} [\xi, s_{E_f}(\xi))$.
- Take any $\xi \in E_2 \backslash \delta$. Then, $\delta \leq \xi < s_A(\xi) \in A$ and since $A \cap [\xi, s_{E_f}(\xi)) = \emptyset$, we get $s_{E_f}(\xi) \leq s_A(\xi)$.
- However $s_{E_f}(\xi) \in E_f$ and so is closed under f. Then $f(\xi) < s_{E_f}(\xi) \leq s_A(\xi)$ and so $f \leq_{cl} s_A$.
- Therefore $\{s_A : A \in F\}$ is $\leq_{cl}$-dominating, and so $\mathfrak{d}_{cl}(\kappa) \leq |F| = \mathfrak{r}(\kappa)$.
- Since $\kappa \geq \beth_\omega$, $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$ and so $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$. □

Strong Unsplitting: $\mathfrak{r}_\sigma(\kappa)$

Definition

$\mathfrak{r}_\sigma(\kappa)$ is the least size of a $F \subseteq [\kappa]^\kappa$ such that there is no countable $\{B_n : n \in \omega\} \subseteq [\kappa]^\kappa$ such that every $A \in F$ is split by some B_n.

Remark

If $\mathfrak{r}_\sigma(\kappa)$ exists, then $\mathfrak{r}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$.

Theorem (Zapletal)

If $\aleph_0 < \kappa \leq 2^{\aleph_0}$ then there is a countable $\mathscr{B}$ splitting all $A \in [\kappa]^\kappa$.

$\mathfrak{d}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$

Remark

Thus $\mathfrak{r}_\sigma(\kappa)$ does not exist for $\aleph_0 < \kappa \leq 2^{\aleph_0}$. However:

Theorem(F., D. Soukup, 2018)

If $\kappa > 2^{\aleph_0}$ is regular, then $\mathfrak{r}_\sigma(\kappa)$-exists and $\mathfrak{d}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$.

$\mathfrak{d}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$

- Take $F \subseteq [\kappa]^\kappa$ of size $\mathfrak{r}_\sigma(\kappa)$, which is unsplit by countable $\mathscr{A} \subseteq [\kappa]^\kappa$.
- If F does not have the RS-property, then $\mathfrak{d}(\kappa) \leq |F| = \mathfrak{r}_\sigma(\kappa)$.
- Thus suppose F has the *RS*-property. That is, for every club E_1 there is a club $E_2 \subseteq E_1$ so that for every $A \in F$,

$$A \not\subseteq^* \bigcup_{\xi \in E_2} [\xi, s_{E_1}(\xi)).$$

- We will show that $\{s_A : A \in F\}$ is dominating.

$\mathfrak{d}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$

- Let $f \in {}^\kappa\kappa$. Wlg f is non-decreasing. Take E_0 a club of ordinals closed under f.
- Applying the RS-property inductively, build a sequence

$$E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$$

  of clubs such that for all $A \in F$ and $n \in \omega$

$$A \not\subseteq^* B_n = \bigcup_{\xi \in E_{n+1}} [\xi, s_{E_n}(\xi)).$$

- Since F is a witness to $\mathfrak{r}_\sigma(\kappa)$, $\{B_n\}_{n\in\omega}$ do not split F and so
- $\exists A \in F$ unsplit by all B_n's. Thus $A \cap B_n$ is bounded for each n.

$\mathfrak{d}(\kappa) \leq \mathfrak{r}_\sigma(\kappa)$

- Thus for each n, there is δ_n such that $A \backslash \delta_n \subseteq \kappa \backslash B_n$, and so
- for $\delta = \sup \delta_n$ we have that for all n, $A \backslash \delta \subseteq \kappa \backslash B_n$.
- Take any $\alpha \in \kappa \backslash \delta$ and let $\xi_n = \sup(E_n \cap (\alpha + 1))$, for each n.
- Then $\{\xi_n\}_{n \in \omega}$ is decreasing, and so there is $n \in \omega$, such that $\xi_n = \xi_{n+1} = \xi$ for some ξ.
- Thus $\xi \leq \alpha$ and $\xi \in E_{n+1} \subseteq E_n$.

- Now $\xi \leq \alpha < s_{E_n}(\alpha)$ and since $E_n \subseteq E_0$, $s_{E_n}(\alpha)$ is closed under f. Thus
$$\xi \leq \alpha \leq f(\alpha) < s_{E_n}(\xi).$$
- On the other hand $\alpha \geq \delta$, $s_A(\alpha) \in A$ and so
$$s_A(\alpha) \notin \bigcup_{\zeta \in E_{n+1}} [\zeta, s_{E_n}(\zeta)).$$
- Therefore, $s_A(\alpha) \notin [\xi, s_{E_n}(\xi))$ and so $s_{E_n}(\xi) \leq s_A(\alpha)$.
- Thus $f(\alpha) < s_{E_n}(\xi) \leq s_A(\alpha)$. □

Thus $\{s_A : A \in F\}$ is indeed $\leq^*$-dominating.

Characterizing $\mathfrak{d}(\kappa)$

Among others, the above result leads to a new characterization of $\mathfrak{d}(\kappa)$ for regular uncountable κ.

Finitely Unsplitting Number

Definition

Let $\mathfrak{fr}$ denote the minimal size of a family $\mathscr{I}$ of partitions of ω into finite sets, so that there is no single $A \in [\omega]^\omega$ with the property that for each partition $\{I_n\}_{n\in\omega} \in \mathscr{I}$ both

$$\{n \in \omega : I_n \subseteq A\} \text{ and } \{n \in \omega : I_n \cap A = \emptyset\}$$

are infinite. That is, there is no A, which interval-splits all partitions.

Theorem (Brendle)

$\mathfrak{fr} = \min\{\mathfrak{d}, \mathfrak{r}\}$.

Higher analogues: $\mathfrak{fr}(\kappa)$, Club unsplitting number

Definition

For κ regular uncountable, let $\mathfrak{fr}(\kappa)$ denote the minimal size of a family $\mathscr{E}$ of clubs, so that there is no $A \in [\kappa]^\kappa$ such that for all $E \in \mathscr{E}$ both

$$\{\xi \in E : [\xi, s_E(\xi)) \subseteq A\} \text{ and } \{\xi \in E : [\xi, s_E(\xi)) \cap A = \emptyset\}$$

have size κ. That is, there is no A, which interval-splits all clubs E.

Higher analogues: $\mathfrak{fr}_\sigma(\kappa)$, Strong club unsplitting

Definition

For κ regular uncountable, let $\mathfrak{fr}_\sigma(\kappa)$ denote the minimal size of a family $\mathscr{E}$ of clubs so that there is no countable $\mathscr{A} \subseteq [\kappa]^\kappa$ with the property that every $E \in \mathscr{E}$ is interval-split by a member of $\mathscr{A}$.

Higher analogues: $\mathfrak{fr}_\sigma(\kappa)$, Strong club unsplitting

Definition

For κ regular uncountable, let $\mathfrak{fr}_\sigma(\kappa)$ denote the minimal size of a family $\mathscr{E}$ of clubs so that there is no countable $\mathscr{A} \subseteq [\kappa]^\kappa$ with the property that every $E \in \mathscr{E}$ is interval-split by a member of $\mathscr{A}$. That is, there is no countable $\mathscr{A} \subseteq [\kappa]^\kappa$ with the property that for each $E \in \mathscr{E}$ there is $A \in \mathscr{A}$ with the property that both

$$\{\xi \in E : [\xi, s_E(\xi)) \subseteq A\} \text{ and } \{\xi \in E : [\xi, s_E(\xi)) \cap A = \emptyset\}$$

have size κ.

Characterization of $\mathfrak{d}(\kappa)$: Strong club unsplitting

Theorem (F., D. Soukup, 2018)

Let κ be a regular uncountable. Then $\mathfrak{d}(\kappa) = \mathfrak{r}_\sigma(\kappa)$.

On cofinalities

Remark

It is a long-standing open problem if $\mathfrak{r}$ can be of countable cofinality. However, if $\mathrm{cf}(\mathfrak{r}) = \omega$ then $\mathfrak{d} \leq \mathfrak{r}$.

Theorem (F., Soukup, 2018)

If κ is regular, uncountable and $\mathrm{cf}(\mathfrak{r}(\kappa)) \leq \kappa$ then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$.

Questions

- (Cummings-Shelah) Does $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathrm{cl}}(\kappa)$ for all regular uncountable κ?
- (Raghavan-Shelah) Does $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ for all regular uncountable κ?

Questions

- (Cummings-Shelah) Does $\mathfrak{d}(\kappa) = \mathfrak{d}_{\mathsf{cl}}(\kappa)$ for all regular uncountable κ?
- (Raghavan-Shelah) Does $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$ for all regular uncountable κ?

Thank you!