Philosophical implications of the paradigm shift in model theory

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Book

Model Theory and the Philosophy of Mathematical Practice:

Formalization without Foundationalism

The book is both a case study of one area of mathematics, model theory, and an argument that developments in that area have more general philosophical interest.

What paradigm shift?

Before

The paradigm around 1950 concerned the study of logics; the principal results were completeness, compactness, interpolation and joint consistency theorems.

Various semantic properties of theories were given syntactic characterizations but there was no notion of partitioning all theories by a family of properties.

What paradigm shift?

After

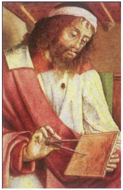
After the paradigm shift there is a systematic search for a finite set of syntactic conditions which divide first order theories into disjoint classes such that models of different theories in the same class have similar mathematical properties.

In this framework one can compare different areas of mathematics by checking where theories formalizing them lie in the classification.

Section I.

Axiomatization vrs Formalization

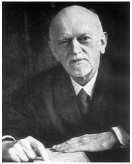
Euclid-Hilbert formalization 1900:

Euclid  Hilbert 

The Euclid-Hilbert (the Hilbert of the Grundlagen der Geometrie) framework has the notions of axioms, definitions, proofs and, with Hilbert, models.

But the arguments and statements take place in natural language. For Euclid-Hilbert logic is a means of proof.

Hilbert-Gödel-Tarski formalization 1917-1956:

Hilbert  Gödel  Tarski 

In the Hilbert (the founder of proof theory)-Gödel-Tarski framework, logic is a mathematical subject.

There are explicit rules for defining a formal language and proof. Semantics is defined set-theoretically.

Bourbaki on Axiomatization:

Dieudonne  Bourbaki  Cartan 

Bourbaki wrote:

> *'We emphasize that it [formalization] is but one aspect of this [the axiomatic] method, indeed the least interesting one.'*

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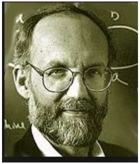
> *'We emphasize that it [formalization] is but one aspect of this [the axiomatic] method, indeed the least interesting one.'*

We reverse Bourbaki's aphorism to argue.

Full formalization is an important tool for modern mathematics.

Formalization

Feferman  Barwise 

Anachronistically, *full formalization* involves the following components.

1. Vocabulary: specification of primitive notions.
2. Logic
   - a Specify a class of well formed formulas.
   - b Specify truth of a formula from this class in a structure.
   - c Specify the notion of a formal deduction for these sentences.
3. Axioms: specify the basic properties of the situation in question by sentences of the logic.

Item 2c) is the least important from our standpoint.

Choice of Logic

This talk focuses on finitary first order logic.
Recent work on infinitary logic (Abstract Elementary Classes):

1. (Boney, Shelah, Vasey) Eventual categoricity from large cardinals
2. (Vasey) A stability hierarchy for tame aec

But where are the examples?

Structures and Definability

A vocabulary τ is collection of constant, relation, and function symbols.

A τ-structure is a set in which each τ-symbol is interpreted.

A subset A of a τ-structure M is definable in M if there is $\mathbf{n} \in M$ and a τ-formula $\phi(x, \mathbf{y})$ such that

$$A = \{m \in M : M \models \phi(m, \mathbf{n})\}.$$

Note that if property is defined without parameters in M, then it is uniformly defined in all models of $\mathrm{Th}(M)$.

Section II. Early Model Theory

What hath Tarski (Robinson?) wrought?

Apparently the first modern statement of the ‘extended completeness theorem’ is in Robinson 1951 *Introduction to Model Theory and to the Metamathematics of Algebra*.

Completeness theorem (modern statement)

For every vocabulary τ and every sentence $\phi \in \mathcal{L}(\tau)$

$$(*) \qquad \Sigma \vdash \phi \text{ if and only if } \Sigma \models \phi.$$

Such a statement assumes Tarski’s definition of truth.

Henkin versus Gödel: proof of the completeness theorem

Henkin 

1. Gödel's definition of 'satisfiability in a structure' depends on the ambient deductive system. Specifically, the deductive system must support the existence of a π_2-prenex normal form for each non-refutable sentence.
   And he doesn't give a formal definition of satisfiability for even π_2 sentences.
2. He extends the vocabulary of the given theory by new relation symbols, Henkin adds only constants.

Two Theses

1. Contemporary model theory makes *formalization* of specific mathematical areas a powerful tool to investigate both mathematical problems and issues in the philosophy of mathematics (e.g. methodology, axiomatization, purity, categoricity and completeness).
2. Contemporary model theory enables systematic comparison of local formalizations for distinct mathematical areas in order to organize and do mathematics, and to analyze mathematical practice.

Theories

Contemporary model theory focuses on theories not logics.

Theories may be given by axioms (first order Peano) or as $\mathrm{Th}(M)$ (true arithmetic).

Examples

algebraically closed fields, dense linear order, the random graph, differentially closed fields, free groups, ZFC,

$\mathrm{Th}(Z, S)$
$\mathrm{Th}(Z, +)$
$\mathrm{Th}(Z, +, 1)$
$\mathrm{Th}(Z, +, 1, \times)$

Classification of Theories

The breakthroughs of model theory as a tool for organizing mathematics come in several steps.

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The breakthroughs of model theory as a tool for organizing mathematics come in several steps.

1. The significance of (complete) first order theories
2. The significance of classes of (complete) first order theories: Quantifier reduction

   'Applied' Quantifier reduction in a natural language is essential for mathematical application.

   'Pure' Quantifier elimination by fiat exposes the fundamental model theoretic structure.
3. stability hierarchy

Model Theory in the 60's: Morley-Vaught

The collection of formulas p is a complete type over A if it satisfies one of the following equivalent conditions.

1. p is a maximal consistent set of formulas $\phi(x, \mathbf{a})$ with parameters from A.
2. p is a member of the Stone Space of the Lindenbaum algebra of A.
3. The solutions of p are an orbit of the group of automorphisms of the monster model which fix A.

Types as descriptions: stability

A complete n-type over the empty set is a description of an n-tuple (over the empty set).

Replace T by $\mathrm{Th}(M, A)$ where $M \models T$ and $A \subset M$.
A complete n-type in $S_n(\mathrm{Th}(M, A)$ is a description of an n-tuple over A.

Definition

Write $S_n(M, A)$ for $S_n(\mathrm{Th}(M, A))$.

The complete theory T is λ-stable if for every $M \models T$ and every $A \subset M$,

$$|A| \leq \lambda \Rightarrow S_n(M, A) \leq \lambda.$$

Semantic classification of first order theories

Theorem

Every countable complete first order theory lies in exactly one of the following classes.

1. (unstable) T is stable in no λ.
2. (strictly stable) T is stable in exactly those λ such that $\lambda^\omega = \lambda$
3. (superstable) T is stable in those $\lambda \geq 2^{\aleph_0}$.
4. (ω-stable) T is stable in all infinite λ.

Syntactic classification of first order theories

Theorem

Every countable complete first order theory lies in exactly one of the following classes.

1. (unstable) T has the order property; some formula $\phi(\mathbf{x}, \mathbf{y})$ defines a linear order on an infinite subset of M^n.
2. (stable) For every formula ϕ, there is an integer n and a formula ϕ_n asserting 'there is no sequence of n-elements with the ϕ-order property'.
3. (superstable) There is a global rank R_C (with respect to n-inconsistency) such that $R_C(\psi) < \infty$ for all ψ.
4. (ω-stable) There is a global rank R_M (with respect to inconsistency) such that $R_M(\psi) < \infty$ for all ψ

From all theories towards classification

Theorem

1. the (strict) hierarchies on the last two slides are the same.
2. The defining conditions are either arithmetic or Π^1_1, so absolute in ZFC.

Historical Consequence

After the paradigm shift first order model theory is no longer so tightly entangled with axiomatic set theory.

Example

Two Cardinal Models

1. A two cardinal model is a structure M with a definable subset D with $\aleph_0 \leq |D| < |M|$.
2. T in a vocabulary with a unary predicate P *admits* (κ, λ) if there is a model M of T with $|M| = \kappa$ and $|P^M| = \lambda$.

Reversing the question

set theorist: For which cardinals $\kappa, \lambda, \kappa', \lambda'$ does
T admits (κ, λ) imply T admits (κ', λ')
for all theories T?

model theorist: For which theories T does
T admits (κ, λ) imply T admits (κ', λ')
for all 4-tuples of cardinals ?

Answer: sufficient: T is stable or o-minimal.

Thesis 4

The study of geometry is not only the source of the idea of axiomatization and many of the fundamental concepts of model theory, but geometry itself (through the medium of geometric stability theory) plays a fundamental role in analyzing the models of tame theories and solving problems in other areas of mathematics.

Three kinds of geometry

1. First order Euclidean geometry
2. first order formalizations of real and complex algebraic geometry
3. combinatorial geometry

What hath Hilbert wrought (elementary geometry)

Hilbert shows that First order axioms for euclidean geometry suffice for polygonal geometry including area and proportion and the basics of circle including right angle trigonometry.

I show that by adding a constant for π one can justify in first order logic the formulas for area and circumference of a circle.

These theories are constructively consistent (in PRA).

Hilbert uses the Archimedean and Dedekind axioms only for:

1. metamathematical investigations
2. asserting the identity of the plane satisfying all the axioms with geometry over the reals.

$\aleph_1$-categorical theories

Morley  Lachlan  Zilber 

Theorem

A complete theory T is strongly minimal if and only if it has infinite models and

1. algebraic closure induces a pregeometry on models of T;
2. any bijection between *acl*-bases for models of T extends to an isomorphism of the models

These two conditions assign a unique dimension which determines each model of T.
Strongly minimal sets are the building blocks of structures whose first order theories are categorical in uncountable power.

The role of geometry

The ability to ascribe dimension is the essence of combinatorial geometry.
If T is a stable theory then there is a notion 'non-forking independence' which has major properties of an independence notion in the sense of van den Waerden.

It imposes a dimension on the realizations of regular types.

For many models of appropriate stable theories it assigns a dimension to the model.

This is the key to being able to describe structures.

Shelah classification strategy 

A property P is a dividing line if
both P and $\neg P$ are virtuous — have significant mathematical consequences.

Stable and superstable are dividing lines

ω-stable and $\aleph_1$-categorical are virtuous but not dividing lines.

Which theories are well behaved

The Main Gap: T has many models or is 'controlled by the countable'

Let T be a countable complete first order theory.

1. Either $I(T, \aleph_\alpha) = 2^{\aleph_\alpha}$ or
2. T is superstable without the *omitting types order property* or the *dimensional order property* and is shallow whence
    1. each model of cardinality λ is decomposed into countable models indexed by a tree of countable height and width λ.
    2. and thus, for any ordinal $\alpha > 0$, $I(T, \aleph_\alpha) < \beth_\delta(|\alpha|)$ (for a countable ordinal δ depending on T);

Either there is uniform way to assign invariants or there is the maximal number of models in every uncountable power.

It's inevitable: Abstract Model theory to algebra

Hart, Hrushovski, Laskowski

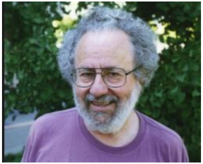
Any model of a complete theory, whose uncountable spectrum is

$$I(\aleph_\alpha, T) = \min\left(2^{\aleph_\alpha}, \beth_{d-1}(|\alpha + \omega| + \beth_2)\right)$$

for some finite $d > 1$,

interprets an infinite group.

Can infinity be tamed? Davis



Martin Davis wrote:

Gödel showed us that the wild infinite could not really be separated from the tame mathematical world where most mathematicians may prefer to pitch their tents.

No! We systematically make this separation in important cases. What ”Gödel showed us is that the wild infinite could not really be separated from the tame mathematical world if we insist on starting with the wild worlds of arithmetic or set theory.

The crucial contrast is between:
a foundationalist approach – demand global foundations
and a foundational approach – search for mathematically important foundations of different topics.

Two ways to study analysis

1. definable analyis: e.g. 0-minimality (with applications to number theory)
2. axiomatic analysis: e.g. differentially closed fields $(F, +, *, 0, 1, d/dx)$

Axiomatic Analysis: Example

Fuchsian differential equation

$$S_{\frac{d}{dt}}(y) + (y')^2 R_\Gamma(y) = 0$$

In his famous ‘Leçons de Stockholm’, Painlevé conjectured that over any differential field extension K of $\mathcal{C}(t)$,

$$\mathrm{tr.deg} K(y, y', y'') = 0 \text{ or } 3.$$

Nagloo discovered that irreducibility in this setting is equivalent to a general model theoretic property called strong minimality.

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Theorem (Casale-Freitag-Nagloo)

Over any differential field extension K of $\mathcal{C}(t)$,

$$\text{tr. deg} K(y, y', y'') = 0 \text{ or } 3.$$

Previously, only partial results, most notably most notably work of Nishioka from the 1970s and 1980s.

Reliability or Clarity

. . . a long-term look at achievements in mathematics shows that genuine mathematical achievement consists primarily in making clear by using new concepts *. . .*

We look for uses of mathematical logic in bringing out these roles of of concepts in mathematics. Representations and methods from the reliability programs are not always appropriate.

We need to be able to emphasize special *features of a given mathematical area and its relationship to others, rather than how it fits into an absolutely general pattern. (Manders 1987)*

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More precisely,
Mathematical logic is tool to solve not only its own problems but to organize and do traditional mathematics.