Purity, content, and arithmetic

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Fermat's Last Theorem is just about numbers, so it seems like we ought to be able to prove it by just talking about numbers.

M. Nathanson, *Elementary Methods in Number Theory* (2000)

The theorems in this book are simple statements about integers, but the standard proofs require contour integration, modular functions, estimates of exponential sums, and other tools of complex analysis. This is not unfair. In mathematics, when we want to prove a theorem, we may use any method. The rule is "no holds barred." It is OK to use complex variables, algebraic geometry, cohomology theory, and the kitchen sink to obtain a proof. But once a theorem is proved, once we know that it is true, particularly if it is a simply stated and easily understood fact about the natural numbers, then we may want to find another proof, one that uses only "elementary arguments" from number theory. Elementary proofs are not better than other proofs, nor are they necessarily easy. Indeed, they are often technically difficult, but they do satisfy the aesthetic boundary condition that they use only arithmetic arguments.

During the nineteenth century, algebraic geometers became interested in the following question: given an algebraic surface, characterize the families of surfaces that intersect the given surface in curves of particular kinds.

Geometers made progress on the case when these families of surfaces formed a kind of linearly dependent system.

In 1905 Enriques gave a complete characterization of such families of surfaces intersecting a given smooth algebraic surface in an irreducible and continuous system of curves.

Castelnuovo and Enriques, appendix to *Théorie des fonctions algébriques de deux variables indépendantes* by E. Picard and G. Simart (1906)

We have not succeeded in demonstrating this theorem using geometric methods…This result is therefore the fruit of a long series of researches, to which the transcendental methods of M. Picard and the geometrical methods used in Italy contributed equally.

In fact, Enriques had not succeeded even in that; his proof was wrong, as discovered by Severi in 1921, who then gave a new proof that was quickly seen to be incorrect.

A correct proof, entirely transcendental, had in the meantime been given by Poincaré.

Castelnuovo, letter to Segre, December 18, 1945

We would need to have a fully satisfying geometrical demonstration.

A. Brigaglia and C. Cilberto, *Italian algebraic geometry between the two world wars* (1995)

In fact, for Enriques and Severi, who were postulating a central role for the projective algebro-geometric methods in mathematics, the missing resolution of such an essential problem in the theory of surfaces—which they considered as their creation—was always an unacceptable humiliation. Thus they repeatedly returned to the consideration of this problem, even if they never were definitively successful in resolving it.

In 1966 D. Mumford gave a purely algebraic-geometric proof of the result, using methods developed by Serre, Grothendieck, and Kodaira.

Briançon-Skoda Theorem (1974)

Let R be either the formal or convergent power series ring in d variables and let I be an ideal of R. Then $\overline{I^d} \subseteq I$, where $\overline{I}$ is the *integral closure* of an ideal I.

Lipman & Teissier, "Pseudo-rational local rings and a theorem of Briançon-Skoda about integral closures of ideals", *Michigan Mathematical Journal* (1981)

The proof given by Briançon and Skoda of this completely algebraic statement is based on a quite transcendental deep result of Skoda.... The absence of an algebraic proof has been for algebraists something of a scandal—perhaps even an insult—and certainly a challenge.

Lipman & Tessier then give such an algebraic proof.

What is this “scandal", this “challenge"?

It is common nowadays to formulate the issues raised here in terms of purity of methods.

Roughly, a solution to a problem, or a proof of a theorem, is pure if it draws only on what is “close" or “intrinsic” to that problem or theorem.

Other common language: avoids what is “extrinsic", “extraneous", “distant", “remote", “alien" or “foreign" to the problem or theorem.

Aristotle, *Posterior Analytics*

We cannot, in demonstrating, pass from one kind to another. We cannot, for instance, prove geometrical truths by arithmetic.

After noting that the planar Desargues theorem is typically proved using space, Hilbert remarks:

Hilbert, “Lectures on Euclidean Geometry”, 1898–1899

Therefore we are for the first time in a position to put into practice *a critique of means of proof*. In modern mathematics such criticism is raised very often, where the aim is to preserve *the purity of method*, i.e. to prove theorems if possible using means that are suggested by the content [*Inhalt*] of the theorem.

Call everything that belongs to the content of a statement, the topic of that statement: definitions, axioms, inferences, etc.

A proof of a theorem is topically pure when it draws only on what belongs to the theorem's topic.

For example, it doesn't seem that set theory belongs to the topic of FLT.

Hilbert later revised the connection between purity and content.

He came to describe some parts of mathematics as being contentual, and others not, but rather ideal.

Non contentual mathematics includes complex numbers, projective points, and infinite sets.

While it is in practice necessary to use non contentual methods in order to simplify our reasoning (“the boxer’s gloves”), only contentual reasoning provides for knowledge.

Thus everything provable by contentual methods must also be provable by contentual reasoning.

Indeed, that non contentual reasoning is always eliminable must itself be proved by contentual reasoning: this is where Gödelian problems intercede.

Hilbert & Bernays, *Grundlagen der Mathematik*, Vol. 1 (1934)

Our treatment of the basics of number theory and algebra was meant to demonstrate how to apply and implement direct contentual inference that takes place in thought experiments on intuitively conceived objects and is free of axiomatic assumptions. Let us call this kind of inference “finitist” inference for short, and likewise the methodological attitude underlying this kind of inference as the “finitist” attitude or the “finitist” standpoint.. . . With each use of the word “finitist”, we convey the idea that the relevant consideration, assertion, or definition is confined to objects that are conceivable in principle, and processes that can be effectively executed in principle, and thus it remains within the scope of a concrete treatment.

Hilbert, "Die Grundlagen Der Elementaren Zahlentheorie" (1931)

This is the fundamental mode of thought that I hold to be necessary for mathematics and for all scientific thought, understanding, and communication, and without which mental activity is not possible at all.

Tait, "Finitism" (1981)

[Finitistically acceptable reasoning] is a minimal kind of reasoning presupposed by all non-trivial mathematical reasoning about numbers.

Infinitude of primes (IP). For every natural number, there is a greater prime number.

Euclidean solution: if $a = 1$, then since $2 = S(1)$ is prime, we know that there is a prime greater than $a = 1$. So suppose that $a > 1$. Let $p_1, p_2, \ldots, p_n$ be all the primes less than or equal to a, and let $Q = S(p_1 \cdot p_2 \cdots p_n)$. Then Q has a prime divisor b that is not equal to any of the p_i, and so $b > a$.

Is it pure?

Finitary arithmetic (PRA , via Tait). Euclidean proof works.

Feasible arithmetic try 1 ($I\Delta_0$). Open problem whether IP can be proved purely; Euclidean proof fails since $I\Delta_0$ does not prove that every product of primes exists.

Feasible arithmetic try 2 (EFA=$I\Delta_0(exp)$). Euclidean solution works.

Feasible arithmetic try 3 ($I\Delta_0 + PHP$, via Paris, Wilkie, Woods). Weaker than EFA. IP is provable here by a different proof, due to Sylvester.

The finitary / infinitary divide is one way to see purity in number theory.

Another is the use of complex analysis.

This divide between the “elementary” and not arose with the complex analytic proofs of the prime number theorem by Hadamard and de la Vallée Poussin in 1896.

The prime number theorem: the number of primes up to n is approximately $\frac{n}{\log n}$.

Ingham, *The Distribution of Prime Numbers*, 1932

> The solution just outlined may be held to be unsatisfactory in that it introduces ideas very remote from the original problem, and it is natural to ask for a proof of the prime number theorem not depending on the theory of a complex variable.

In 1949, Selberg and Erdős independently found such proofs (a chief reason for Selbert's Fields Medal).

Takeuti showed that the proofs of classical analytic number theory can be carried out in a formalization of complex analysis he calls elementary complex analysis (ECA).

In particular the PNT can be expressed in $\mathcal{L}_{ECA}$, and the HdlVP proof can be formalized in ECA.

Takeuti (1978): ECA is a conservative extension of PA.

Thus $\mathrm{PNT}_{\mathrm{PA}}$ is a theorem of PA.

Sudac (2001): the HdlVP proof can be formalized in $\mathrm{I}\Sigma_1$.

Cornaros and Dimitracopoulos (1994) formalized Selberg's proof in EFA.

Ingham (1932): “an argument which makes no explicit mention of analytic functions may nevertheless involve closely related ideas”.

The PNT had been shown equivalent to the non-existence of zeros of the Riemann zeta function on the line $\mathrm{Re}(s) = 1$.

Granville, “Analytic Number Theory” in *The Princeton Companion to Mathematics*, 2008

One might argue that it is inevitable that complex analysis must be involved [in a proof of the PNT]. . . . Of course [Selberg and Erdős’] proof must somehow show that there is no zero on the line $Re(s) = 1$, and indeed their combinatorics cunningly masks a subtle complex analysis proof beneath the surface.

This raises subtle questions about “tacit” or “hidden” complex-analytic content of arithmetical sentences.

Here are some reasons to value purity.

- Comprehension requires minimal resources
- Transmission requires minimal resources
- It induces a smaller search space
- Searching for pure proofs helps discipline or train the mind
- Knowing more about a single subject puts us in a better position for learning more about it later
- There is a natural order of truths, and pure proofs best track that order.
- Pure proofs explain their theorems
- Pure proofs provide for enduring, stable knowledge of their conclusions.

The desire for purity and the desire for impurity would only be in conflict if we could only give one proof of each theorem.

Rather, we should see purity and impurity as distinct epistemic values in mathematics.

It is important to cultivate a plurality of epistemic values in order to succeed as a mathematical knower, because to know requires knowing in as many different ways as we can.

Nietzsche, *On the Genealogy of Morals*

There is only a perspectival seeing, only a perspectival "knowing"; and the more affects we allow to speak about a matter, the more eyes, different eyes, we know how to bring to bear on one and the same matter, that much more complete will our "concept" of this matter, our "objectivity" be.