$I_0(\lambda)$ and Combinatorics at λ^+

Xianghui Shi
Beijing Normal University

The 4^{th} Arctic Set Theory Workshop
January 21-26, 2019 @ Kilpisjärvi

This is a joint work with Nam Trang.

Axiom I_0

Definition

Axiom $I_0(\lambda)$ is the assertion that there is a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ such that $\mathrm{crit}(j) < \lambda$.

- It was first proposed and studied by Woodin in the early 80's and by Laver in the 90's.
- It is by far (among) the strongest (in terms of consistency strength) large cardinal axioms unknown to be inconsistent with ZFC.
- Write $I_0(\lambda, X, \alpha)$ for the relativized (to an $X \subseteq V_{\lambda+1}$) version: "there is a $j : L_\alpha(X, V_{\lambda+1}) \prec L_\alpha(X, V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$".

Supercompact

Definition

- κ is λ-supercompact if there is an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and ${}^{\lambda}M \subseteq M$.
- κ is supercompact if it is λ-supercompact for every $\lambda \geq \kappa$.

- Supercompactness implies the consistency of most forcing axioms.
- If $I_0(\lambda)$ holds, then λ is a limit of very strong large cardinals, for instance, limit of $<\lambda$-supercompact cardinals.
- Although the statement $I_0(\lambda)$ is stronger than the existence of supercompact cardinals in terms of consistency strength, what it directly implies is not very much beyond the existence of $<\lambda$-supercompact cardinals.
- There are a fair number of statements that follow from supercompactness but are independent of $I_0(\lambda)$.

Three types of questions

Let φ be a combinatorical principle at λ^+. In this talk, we look into the compatibility of $I_0(\lambda)$ with various φ's over the base theory $\Gamma = \mathsf{ZFC} + I_0(\lambda)$. For each φ, we ask three questions:

- Is φ consistent with Γ?
- Is $\neg\varphi$ consistent with Γ?
- Is φ true in $L(V_{\lambda+1})$?

Combinatorial Principles

The combinatorial principles discussed in this talk include

1. the existences of (special) λ^+-Aronszajn tree and of λ^+-Suslin tree;
2. the $\square_\lambda$ and the $\square^*_\lambda$ principles;
3. the existence of (good, very good) scale at λ^+;
4. stationary reflection at λ^+;
5. the $\diamondsuit_{\lambda^+}$ principle;
6. GCH (as well as SCH) at λ.

λ^+-Aronszajn tree

Definition

- κ-tree is a tree on κ of size κ whose every level has size $<\kappa$.
- A κ-Aronszajn tree is a κ-tree that has no cofinal branch of length κ.
- A κ-Aronszajn tree is special if it is union of κ-many antichains.

λ^+-Aronszajn tree

Definition

- κ-tree is a tree on κ of size κ whose every level has size $<\kappa$.
- A κ-Aronszajn tree is a κ-tree that has no cofinal branch of length κ.
- A κ-Aronszajn tree is special if it is union of κ-many antichains.

Theorem 1

Assume ZFC $+ I_0(\lambda)$. *There is no* λ^+ *Aronszajn tree in* $L(V_{\lambda+1})$.

Proof.

- $I_0(\lambda)$ implies that $L(V_{\lambda+1}) \models \lambda^+$ is a measurable cardinal.
- Assume towards a contradiction that T is a λ^+-Aronszajn tree in $L(V_{\lambda+1})$.
- Let $\pi : L[T] \to M \cong \text{Ult}(L[T], \mu \cap L[T])$ be the ultrapower embedding induced by a λ^+-complete measure μ on λ^+. Then $\pi(T)$ is a $\pi(\lambda^+)$-Aronszajn tree in M.
- Since $\mathrm{crit}(\pi) = \lambda^+$, we have $T = \pi``T \subset \pi(T)$ and $\pi(\lambda^+) > \lambda^+$.
- Any node at the λ^+-th level of $\pi(T)$ is a cofinal branch of $\pi``T = T$. Contradiction! □

Square Principle

Definition (Jensen-Schimmerling)

Let λ be an uncountable cardinal. A $\square_{\kappa,\lambda}$-sequence is sequence $\langle \mathcal{C}_\alpha : \alpha \in \lim(\lambda^+)\rangle$ such that for all $\alpha < \lambda^+$,

1. each $\mathcal{C}_\alpha$ is a nonempty set of club subsets of α, $1 \le |\mathcal{C}_\alpha| \le \kappa$;
2. for all $\alpha \in \lim(\lambda^+)$, all $C \in \mathcal{C}_\alpha$ and all $\beta \in \lim(C)$, $\mathrm{otp}(C) \le \lambda$ and $C \cap \beta \in \mathcal{C}_\beta$.

- The classical Jensen's "square principle", $\square_\lambda$, states that there exists a $\square_{1,\lambda}$-sequence, and
- The "weak square" principle, $\square^*_\lambda$, states the existence of a $\square_{\lambda,\lambda}$-sequence.
- Note that $\square^*_\lambda$ is equivalent to the existence of a special λ^+-Aronszajn tree. (Jensen)

Failure of square in $L(V_{\lambda+1})$

A similar argument gives

Theorem 2

Assume ZFC $+ I_0(\lambda)$. *Then* $L(V_{\lambda+1}) \models \neg\Box_\lambda$.

Failure of square in $L(V_{\lambda+1})$

A similar argument gives

Theorem 2

Assume ZFC $+ I_0(\lambda)$. *Then* $L(V_{\lambda+1}) \models \neg\Box_\lambda$.

Remark

Although $\Box_\lambda$ implies the existence of a λ^+-Aronszajn tree, this does not enable us to conclude $L(V_{\lambda+1}) \models \neg\Box_\lambda$ immediately from Theorem 1, as the construction of a λ^+-Aronszajn tree uses λ^+-DC, which in general is not true in $L(V_{\lambda+1})$.

Independence results

Theorem 3 (ZFC)

1. *Assume* $I_0(\lambda)$*. Then there is a model in which* $I_0(\lambda)$ *holds and there is a special* λ^+*-Aronszajn tree, even furthermore a* λ^+*-Suslin tree.*
2. *Assume* $I_0(\lambda, V^\sharp_{\lambda+1}, \omega\cdot 2+1)$*, i.e. there is a*

$$j : L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1}) \prec L_{\omega\cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$*. Then there is a* $\bar\lambda < \lambda$ *such that* $I_0(\bar\lambda)$ *holds and there is no* $\bar\lambda^+$*-Aronszajn tree.*

Independence results

Theorem 3 (ZFC)

1. *Assume* $I_0(\lambda)$. *Then there is a model in which* $I_0(\lambda)$ *holds and there is a special* λ^+*-Aronszajn tree, even furthermore a* λ^+*-Suslin tree.*
2. *Assume* $I_0(\lambda, V^\sharp_{\lambda+1}, \omega \cdot 2 + 1)$, *i.e. there is a*
$$j : L_{\omega \cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1}) \prec L_{\omega \cdot 2+1}(V^\sharp_{\lambda+1}, V_{\lambda+1})$$
with $\mathrm{crit}(j) < \lambda$. *Then there is a* $\bar{\lambda} < \lambda$ *such that* $I_0(\bar{\lambda})$ *holds and there is no* $\bar{\lambda}^+$*-Aronszajn tree.*

The hypothesis in 2, by a theorem of Cramer, implies $I_0(\bar{\lambda})$, for some $\bar{\lambda} < \lambda$.

Theorem 4 (ZFC)

1. $\mathrm{Con}(I_0(\lambda))$ *implies* $\mathrm{Con}(I_0(\lambda) + \square_\lambda)$.
2. *Assume* $I_0(\lambda, V^\sharp_{\lambda+1}, \omega \cdot 2 + 1)$. *Then there is a* $\bar{\lambda} < \lambda$ *such that* $I_0(\bar{\lambda})$ *holds and* $\square_{\bar{\lambda}}$ *fails.*

Scales

- Consider $\prod_{i<\omega} \kappa_i$, where each κ_i is regular and $\lambda = \sup_{i<\omega} \kappa_i$.
- Let $I = \mathrm{Fin}$, i.e. the ideal consisting of all finite subsets of ω.
- Given $f, g \in \prod_i \kappa_i$, $f <_I g$ iff $\omega \setminus \{i \mid f(i) < g(i)\} \in I$.
- A sequence $\langle f_i : i < \alpha \rangle$ is a scale of length α in $\prod_i \kappa_i / I$ if it is $<_I$-increasing and cofinal in $\prod_i \kappa_i / I$.
- A scale for λ is a pair $(\bar{\kappa}, \bar{f})$, where $\bar{f}$ is a scale of length λ^+ in $\prod_i \kappa_i / I$.
- **ZFC-Fact**: There exists a scale for λ whenever λ is singular.

Definition

- Suppose $(\bar{\kappa}, \bar{f})$ is a scale for λ. A point $\alpha < \lambda^+$ is good for $(\bar{\kappa}, \bar{f})$ iff there is an unbounded $A \subset \alpha$ s.t. $\langle f_\beta(n) : \beta \in A \rangle$ is strictly increasing for sufficiently large n.
- α is very good for $(\bar{\kappa}, \bar{f})$ if A above is a club in α.
- A scale $(\bar{\kappa}, \bar{f})$ for λ is good if it is good at every point in $\lambda^+ \cap \mathrm{Cof}({>}\omega)$.
- A scale $(\bar{\kappa}, \bar{f})$ for λ is very good if it is very good at every point in $\lambda^+ \cap \mathrm{Cof}({>}\omega)$.

Theorem 5 (ZFC)

1. *Assume* $I_0(\lambda)$. *There is no scale at* λ *in* $L(V_{\lambda+1})$.
2. *Assume* $I_0(\lambda)$. *Then there is a model of* ZFC $+ I_0(\lambda)$, *in which there is a very good scale at* λ.
3. *Assume* $I_0(\lambda, V^\sharp_{\lambda+1}, \omega \cdot 2 + 1)$. *Then there is a* $\bar\lambda < \lambda$ *such that* $I_0(\bar\lambda)$ *holds and there is no good scale at* $\bar\lambda$.

Singular limit above supercompacts

Theorem

1. (Magidor-Shelah[1996]). If μ is a singular limit of μ^+-strongly compact cardinals, then there is no μ^+-Aronszajn tree.
2. (Solovay[1978] [supercompact], Gregory[strongly compact], Jensen[subcompact], Brooke-Taylor and Sy Friedman[2012]).
   If κ is μ^+-subcompact and $\mu \geq \kappa$, then $\neg\Box_\mu$.
3. (Shelah[1979] [strongly compact], Brooke-Taylor and Sy Friedman[2012]).
   If κ is μ^+-subcompact and $\mathrm{cf}(\mu) < \kappa < \mu$, then $\neg\Box^*_\mu$.
4. (Shelah[1979]). If κ is μ^+-supercompact and $\mathrm{cf}(\mu) < \kappa < \mu$, then there are scales of length μ^+ but none of them are good.

Singular limit above supercompacts

Theorem

1. (Magidor-Shelah^{1996}). If μ is a singular limit of μ^+-strongly compact cardinals, then there is no μ^+-Aronszajn tree.
2. (Solovay$^{1978}_{\text{[supercompact]}}$, Gregory$_{\text{[strongly compact]}}$, Jensen$_{\text{[subcompact]}}$, Brooke-Taylor and Sy Friedman^{2012}). If κ is μ^+-subcompact and $\mu \geq \kappa$, then $\neg\Box_\mu$.
3. (Shelah$^{1979}_{\text{[strongly compact]}}$, Brooke-Taylor and Sy Friedman^{2012}). If κ is μ^+-subcompact and $\mathrm{cf}(\mu) < \kappa < \mu$, then $\neg\Box^*_\mu$.
4. (Shelah^{1979}). If κ is μ^+-supercompact and $\mathrm{cf}(\mu) < \kappa < \mu$, then there are scales of length μ^+ but none of them are good.

- If κ is supercompact, then the hypotheses in (2)-(4) hold at κ.
- The hypotheses in (1)-(4) may fail at $\mu = \lambda$, $\kappa = \mathrm{crit}(j)$, with the presence of $I_0(\lambda)$.

Stationary Reflection

Definition

Let κ be uncountable and regular. Let $S \subseteq \kappa$ be stationary.

- S reflects at α for $\alpha < \kappa$ with $\mathrm{cf}(\alpha) > \omega$ if $S \cap \alpha$ is stationary in α.
- Stationary Reflection Principle for T, where $T \subseteq \kappa$ is stationary, says that for every stationary $S \subseteq T$, S reflects at some $\alpha < \kappa$.
- SRP_{λ^+} denotes the Stationary Reflection Principle for $T = \lambda^+$.

Theorem 6 (ZFC)

1. *Assume* $I_0(\lambda)$ *is consistent. Then so is* $I_0(\lambda) + \neg \mathrm{SRP}_{\lambda^+}$*.*
2. *Assume* $I_0(\lambda, V^\sharp_{\lambda+1}, \omega \cdot 2 + 1)$*. Then there is a* $\bar{\lambda} < \lambda$ *such that* I_0 *holds at* $\bar{\lambda}$ *and* $\mathrm{SRP}_{\bar{\lambda}^+}$ *is true.*

- Due to the lack of choice in this model,[1] the situation of SRP_{λ^+} in $L(V_{\lambda+1})$ is unclear.

[1](Woodin^{1990}). $L(V_{\lambda+1}) \models \mathsf{DC}_{<\lambda^+}(V_{\lambda+1})$.

- We include a scenario where it could be true in $L(V_{\lambda+1})$.

Theorem 7 (ZFC)

Assume $L(V_{\lambda+1}) \models \lambda^+$ *is* $V_{\lambda+1}$*-supercompact, i.e. there is a fine, normal,* λ^+*-complete measure* μ *on* $\mathscr{P}_{\lambda^+}(V_{\lambda+1})$[a][b]*.*
Then $L(V_{\lambda+1}) \models \mathrm{SRP}_{\lambda^+}$*.*

[a] Fineness and completeness have standard meanings.

[b] In the context where full AC does not hold, normality is defined as follows: suppose $F : \mathscr{P}_{\lambda^+}(V_{\lambda+1}) \to \mathscr{P}_{\lambda^+}(V_{\lambda+1})$ is s.t. $\{\sigma : F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset\} \in \mu$, then there is some x such that $\{\sigma : x \in F(\sigma)\} \in \mu$

- We include a scenario where it could be true in $L(V_{\lambda+1})$.

Theorem 7 (ZFC)

Assume $L(V_{\lambda+1}) \models \lambda^+$ is $V_{\lambda+1}$-supercompact, i.e. there is a fine, normal, λ^+-complete measure μ on $\mathscr{P}_{\lambda^+}(V_{\lambda+1})$[a][b].
Then $L(V_{\lambda+1}) \models \mathrm{SRP}_{\lambda^+}$.

---

[a] Fineness and completeness have standard meanings.

[b] In the context where full AC does not hold, normality is defined as follows: suppose $F : \mathscr{P}_{\lambda^+}(V_{\lambda+1}) \to \mathscr{P}_{\lambda^+}(V_{\lambda+1})$ is s.t. $\{\sigma : F(\sigma) \subseteq \sigma \wedge F(\sigma) \neq \emptyset\} \in \mu$, then there is some x such that $\{\sigma : x \in F(\sigma)\} \in \mu$

- However, whether the hypothesis is compatible with $I_0(\lambda)$ is yet unknown.

Sketch of the proof

- Working in $L(V_{\lambda+1})$, fix a measure μ witnessing that λ^+ is $V_{\lambda+1}$-supercompact.
- For each $\sigma \in \mathscr{P}_{\lambda^+}(V_{\lambda+1})$, let $M_\sigma = \mathsf{HOD}_{\sigma \cup \{\sigma\}}$ and let $M = \prod_\sigma M_\sigma / \mu$ be the μ-ultraproduct of the structures M_σ's.
- Łos theorem holds for this ultraproduct.
- Let $S \subseteq \lambda^+$ be stationary and $S^* = [c_S]_\mu$. Then $S^* \cap \lambda^+ = S$ and is stationary (in M). By Łos and the normality of μ, there is some $\alpha < \lambda^+$ such that

$$A = \{\sigma \mid M_\sigma \vDash S \cap \alpha \text{ is stationary}\} \in \mu.$$

- Let $C \subseteq \alpha$ be a club in α. By Łos and the fineness of μ, $B = \{\sigma \mid C \in M_\sigma\} \in \mu$.
- Fix a $\sigma \in A \cap B$. Then in M_σ, C is club in α and $S \cap \alpha$ is stationary, hence $C \cap S \cap \alpha \neq \emptyset$.
- By Łos, $S \cap \alpha$ is stationary.

Definitions from I_0 theory

Definition

Suppose $X \subseteq V_{\lambda+1}$.

1. $\Theta_\lambda^X =_{\mathsf{def}} \{\alpha \mid L(X, V_{\lambda+1}) \models \exists$ a surjective $\pi : V_{\lambda+1} \to \alpha\}$.
2. An ordinal $\alpha < \Theta_\lambda^X$ is X-good if every element of $L_\alpha(X, V_{\lambda+1})$ is definable in $L_\alpha(X, V_{\lambda+1})$ with parameters in $V_{\lambda+1} \cup \{X\}$.

Definitions from I_0 theory

Definition

Suppose $X \subseteq V_{\lambda+1}$.

1. $\Theta^X_\lambda =_{\text{def}} \{\alpha \mid L(X, V_{\lambda+1}) \models \exists$ a surjective $\pi : V_{\lambda+1} \to \alpha\}$.
2. An ordinal $\alpha < \Theta^X_\lambda$ is X-good if every element of $L_\alpha(X, V_{\lambda+1})$ is definable in $L_\alpha(X, V_{\lambda+1})$ with parameters in $V_{\lambda+1} \cup \{X\}$.

- Our discussion regarding the GCH at λ assumes a stronger form of Generic absoluteness.

About proper I_0 embedding

For an $X \subseteq V_{\lambda+1}$, let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ be such that $\mathrm{crit}(j) < \lambda$. Let

$$U = \{X \in L(X, V_{\lambda+1}) \mid j \restriction V_\lambda \in j(X)\}$$

be the ultrafilter given by j. Define

$$W = j(U) = \bigcup \{j(\mathrm{ran}(\pi)) \mid \pi \in L(X, V_{\lambda+1}) \wedge \pi : V_{\lambda+1} \to U\}$$

If j is proper (definition omitted), then

- $j = j_U$.
- W is an $L(X, V_{\lambda+1})$-ultrafilter over $V_{\lambda+1}$, and $\mathrm{Ult}(L(X, V_{\lambda+1}), W)$ is wellfounded.
- $\mathrm{Ult}(L(X, V_{\lambda+1}), W) \cong L(X, V_{\lambda+1})$ and the associated map $j_W : L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ is elementary and proper.
- This process can be iterated, so that for all iterates M_α of $M_0 = L(X, V_{\lambda+1})$ is wellfounded.

Definition (Woodin[2011])

Assume $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ is proper and $\mathrm{crit}(j) < \lambda$. Let $(M_\omega, j_{0,\omega})$ be the ω-iterate of $(L(X, V_{\lambda+1}), j)$. Suppose $\alpha < \Theta^X_\lambda$ and α is X-good. We say that Generic Absoluteness holds for X at α if the following is true:

> *Suppose* $\mathbb{P} \in j_{0,\omega}(V_\lambda)$, $G \in V$ *is an* M_ω*-generic filter for* $\mathbb{P}$, *and* $\mathrm{cf}(\lambda) = \omega$ *in* $M_\omega[G]$. *Then there is some* $\alpha' \leq \alpha$ *and* $X' \subseteq V_{\lambda+1}$ *such that*
>
> $$L_{\alpha'}(X', M_\omega[G] \cap V_{\lambda+1}) \prec L_\alpha(X, V_{\lambda+1}).$$

Definition (Woodin[2011])

Assume $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ is proper and $\mathrm{crit}(j) < \lambda$. Let $(M_\omega, j_{0,\omega})$ be the ω-iterate of $(L(X, V_{\lambda+1}), j)$. Suppose $\alpha < \Theta^X_\lambda$ and α is X-good. We say that Generic Absoluteness holds for X at α if the following is true:

> *Suppose* $\mathbb{P} \in j_{0,\omega}(V_\lambda)$, $G \in V$ *is an* M_ω*-generic filter for* $\mathbb{P}$*, and* $\mathrm{cf}(\lambda) = \omega$ *in* $M_\omega[G]$*. Then there is some* $\alpha' \leq \alpha$ *and* $X' \subseteq V_{\lambda+1}$ *such that*
>
> $$L_{\alpha'}(X', M_\omega[G] \cap V_{\lambda+1}) \prec L_\alpha(X, V_{\lambda+1}).$$

Theorem (Woodin[2011], Cramer[2015])

$I_0(\lambda)$ implies that the Generic Absoluteness for $X = \varnothing$ at all α.

Definition (Woodin[2011])

Assume $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ is proper and $\mathrm{crit}(j) < \lambda$. Let $(M_\omega, j_{0,\omega})$ be the ω-iterate of $(L(X, V_{\lambda+1}), j)$. Suppose $\alpha < \Theta^X_\lambda$ and α is X-good. We say that Generic Absoluteness holds for X at α if the following is true:

> *Suppose* $\mathbb{P} \in j_{0,\omega}(V_\lambda)$, $G \in V$ *is an* M_ω*-generic filter for* $\mathbb{P}$, *and* $\mathrm{cf}(\lambda) = \omega$ *in* $M_\omega[G]$. *Then there is some* $\alpha' \leq \alpha$ *and* $X' \subseteq V_{\lambda+1}$ *such that*
>
> $$L_{\alpha'}(X', M_\omega[G] \cap V_{\lambda+1}) \prec L_\alpha(X, V_{\lambda+1}).$$

Theorem (Woodin[2011], Cramer[2015])

$I_0(\lambda)$ implies that the Generic Absoluteness for $X = \varnothing$ at all α.

- It is unclear for arbitrary X.

Diamond and GCH at λ^+

Theorem 8 (ZFC)

1. *Assume $I_0(\lambda)$. Then in $L(V_{\lambda+1})$, there is no λ^+-sequence of distinct members of $V_{\lambda+1}$, therefore $2^\lambda \neq \lambda^+$ and $\neg\diamondsuit_{\lambda^+}$.*
2. *Assume $\exists\lambda\, I_0(\lambda)$ is consistent. Then so are $\exists\lambda\,(I_0(\lambda) + 2^\lambda = \lambda^+)$ and $\exists\lambda\,(I_0(\lambda) + \diamondsuit_{\lambda^+})$.*
3. (Dimonte-Friedman[2014]). Assume there is a proper

$$j : L(V^\sharp_{\lambda+1}, V_{\lambda+1}) \prec L(V^\sharp_{\lambda+1}, V_{\lambda+1})$$

with $\mathrm{crit}(j) < \lambda$ and $V_\lambda \models$ GCH. Suppose $\alpha \in (\Theta_\lambda, \Theta_\lambda^{V^\sharp_{\lambda+1}})$ and α is $V^\sharp_{\lambda+1}$-good and assume that Generic Absoluteness holds for $V^\sharp_{\lambda+1}$ at α.
Then it is consistent that $I_0(\lambda)$ holds and $2^\lambda > \lambda^+$.

- (1) is an analog of the well-known AD-fact, namely: there is no ω_1-sequence of distinct reals.
- (2) follows from the fact that $\diamondsuit_{\lambda^+}$ can be obtained by forcing $2^\lambda = \lambda^+$ without adding bounded subsets of λ[2], therefore preserves $2^{<\lambda} = \lambda$ and $I_0(\lambda)$.
- For (3), we apply the Generic Absoluteness to Gitik's one-extender-based Prikry forcing, and show that it is λ-good.

  λ-goodness is a sufficient condition, due to Woodin, for a forcing notion $\mathbb{P}$ satisfying the conditions in Generic Absoluteness, i.e. $\mathbb{P} \in j_{0,\omega}(V_\lambda)$ and there is a M_ω-generic filter $G \subset \mathbb{P}$ in V such that $M_\omega[G] \models \mathrm{cf}(\lambda) = \omega$.

  This involves a systematic analysis on the ranks of (finite parts of) conditions in Gitik's forcing.

[2] Use Levy collapse $\mathrm{Coll}(\lambda^+, 2^\lambda)$

Thank you!