Ramsey regularity, MAD families, and their relatives

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Arctic Set Theory 4
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Why there are no analytic MAD families

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Sketch of proof. Suppose that T is tree on $2 \times \omega$ such that

$$\mathcal{A} = p[T]$$

is an a.d. family.

We show $\mathcal{A}$ is not maximal.

An Invariant Tree

For $X \in [\omega]^\omega$ define

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2. T^X is a sub-tree of T,
3. $s \in T^X \iff [T^X_s] \neq \emptyset$ (that is, T^X is pruned),
4. $\emptyset \notin T^X \iff (\forall A \in \mathcal{A})\ A \cap X \in \mathrm{Fin} \iff X$ is a counterexample to maximaliy of $\mathcal{A}$.

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Suppose $s, t \in T^X$, $\mathrm{lh}(s) = \mathrm{lh}(t)$ but $p(s) \neq p(t)$.
Then there are $s' \in T^X_s$ and $t' \in T^X_t$ such that

$$\left(\bigcup p[T^X_{s'}]\right) \cap \left(\bigcup p[T^X_{t'}]\right) \subseteq p(s') \cap p(t').$$

Proof.

Otherwise, we could construct $s = s_0 \sqsubset s_1 \sqsubset \dots$ and $t = t_0 \sqsubset t_1 \sqsubset \dots$ from T such that

$$p\left(\bigcup_{n\in\omega} s_n\right) \cap p\left(\bigcup_{n\in\omega} t_n\right) \notin \mathsf{Fin}$$

which contradicts that $\mathcal{A}$ is an a.d. family. □

The tilde-operator

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Define a map

$$\sim : [\omega]^\omega \to [\omega]^\omega, \\ B \mapsto \tilde{B}$$

by

$$\tilde{B} = \{\hat{A}^l(m) \mid l \in B, m = \min B \setminus (l+1)\}.$$

Properties of the tilde-operator

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Proof of Item 2: Ramsey's Theorem for pairs, or directly using the pigeon hole principle.

The Argument

1. There is $B_0 \in [\omega]^\omega$ and T^* such that

$$(\forall B \in [B_0]^\omega)\ T^{\tilde{B}} = T^*$$

Proof: Using that analytic sets are Ramsey, make $X \mapsto T^{\tilde{X}}$ continuous; by invariance, this map must be constant.

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   Proof: Use the Main Lemma and properties of the tilde operator!

3. In fact $T^* = \emptyset$.

4. Since $\emptyset \notin T^* = T^{\tilde{B}_0}$ it follows that $\tilde{B}_0$ is a counterexample to maximality of $\mathcal{A}$.

‘No MAD families’ from regularity

The previous argument can be generalized to show the following:

Theorem 2

Suppose the following hold:

1. *Dependent Choice (DC),*
2. *Every relation can be uniformized on a Ramsey positive set,*
3. *Every subset of* $[\omega]^\omega$ *is completely Ramsey.*

Then there are no MAD families.

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- There is a projective version of Theorem 2 whose its hypotheses hold after collapsing an inaccessible, or under PD $+$ DC.

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By maximality of $\mathcal{A}$, R is total and so by uniformization we can find $B_0 \in [\omega]^\omega$ and $f\colon [B_0]^\omega \to [\omega]^\omega$ such that

$$(\forall B \in [B_0]^\omega)\ \tilde{B} \cap f(B) \notin \mathsf{Fin} \wedge f(B) \in \mathcal{A}$$

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Then $\mathcal{A}' = \mathrm{ran}(f \restriction [B_0]^\omega)$ is an analytic a.d. family maximal in $\mathrm{ran}(\sim \restriction [B_0]^\omega)$, contradiction.

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One can define Fin^2-MAD families of subsets of ω^2 in the obvious way.

More Theorems

Theorem 3 (Haga-S-Törnquist)

There is no analytic infinite Fin^2*-MAD family.*

Theorem 4

Suppose the following hold:

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As with Theorem 2, there is a 'projective' version of this theorem.

Giitu!