Filters and remainders of topological groups

Arctic Set Theory Workshop 4

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Filters

Given a set X, a *filter* on X is a subset $\mathcal{F} \subset \mathcal{P}(X)$ with the following properties

- $\emptyset \notin \mathcal{F}$,
- $X \in \mathcal{F}$,
- $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
- $A \in \mathcal{F}$ and $A \subset B \subset X$ imply $B \in \mathcal{F}$.

Ultrafilters

An *ultrafilter* (on X) is a filter that is maximal among all filters on X, using the inclusion order.

Filters on X are *free* if they extend the Fréchet filter $\mathfrak{Fr}_X = \{A \subset X : X \setminus A \text{ is finite}\}$.

The existence of free ultrafilters follows from the Axiom of Choice.

Positive sets and ideals

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The *ideal associated* to a filter $\mathcal{F}$ is the set

$$\mathcal{F}^* = \{A \subset X : X \setminus A \in \mathcal{F}\}.$$

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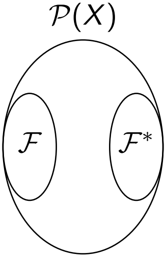

$$\mathcal{F} \subset \mathcal{F}^{+}$$
$$\mathcal{F}^{+} = \mathcal{P}(X) \setminus \mathcal{F}^{*}$$

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Example: ω is a pseudointersection of $\mathfrak{Fr}_\omega$.

A filter $\mathcal{F}$ on X is a P-filter if every $\{A_n : n \in \omega\} \subset \mathcal{F}$ has a pseudointersection $A \in \mathcal{F}$.

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CH implies the existence of P-points on ω.

Theorem (Saharon Shelah, 1978)

There is a model of ZFC with **NO** *P-points on ω.*

Filters as topological spaces

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Thus, $\mathcal{F}$ is a subset of the Cantor set.

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Ultrafilters are non-meager.

The existence of a non-meager P-filters follows from $\mathrm{cof}([\mathfrak{d}]^\omega, \subset) = \mathfrak{d}$. (If all P-filters are meager, then $0^\sharp$ does not exist.)

Countable spaces with one non-isolated point

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Every countable space with a unique non-isolated point is homeomorphic to $\xi(\mathcal{F})$ for some filter $\mathcal{F}$.

The Menger and Hurewicz properties

A space X is *Menger* if every time $\{\mathcal{U}_n : n \in \omega\}$ is a sequence of open covers of X, then for every $n \in \omega$ there is $F_n \in [\mathcal{U}_n]^{<\omega}$ such that $\bigcup\{F_n : n \in \omega\}$ covers X.

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σ compact $\implies$ Hurewicz $\implies$ Menger $\implies$ Lindelöf

Compactifications and Remainders

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Question (Bella, Tokgös, Zdomskyy, 2016)

When is $\beta C_p(X) \setminus C_p(X)$ Menger but not σ-compact?

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Theorem (Marciszewski, 1993)

The following are equivalent for a free filter $\mathcal{F}$ on ω.

(a) $\mathcal{F}$ is a non-meager P-filter.

(b) $\mathcal{F}$ is hereditarily Baire.

(c) $C_p(\xi(\mathcal{F}))$ is hereditarily Baire.

Menger remainders of $C_p(X)$

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Theorem (Bella and HG, 2019?)

Let $\mathcal{F}$ be a free filter on ω. Then $C_p(\xi(\mathcal{F}))$ has a Menger remainder if and only if $\mathcal{F}^+$ is a Menger space (with the topology of the Cantor set).

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Corollary

If there exists a Menger ultrafilter, then there exists a space X such that $\beta C_p(X) \setminus C_p(X)$ Menger but not σ-compact.

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It is consistent with ZFC that there exists a space X such that $\beta C_p(X) \setminus C_p(X)$ Menger but not σ-compact.

Final comments

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Question

Consider the two statements.

(1) *There exists a non-meager P-filter.*

(2) *There is a filter $\mathcal{F}$ with $\mathcal{F}^+$ a Menger filter.*

Does (1) imply (2)? Does the consistency of (1) imply the consistency of (2)?

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2. *(2) There is a filter $\mathcal{F}$ with $\mathcal{F}^+$ a Menger filter.*

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Question

Is there a space X with no isolated points such that $C_p(X)$ has a Menger remainder?

Thank you