Torsion-free abelian groups in (descriptive) set theory

Arctic set theory workshop 4, Kilpisjärvi

Filippo Calderoni
University of Turin

Set theory and abelian groups

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Definition

Let $\boldsymbol{G} = (G, +)$ be an abelian group and κ an infinite cardinal.

- $\boldsymbol{G}$ is **free** if and only $\boldsymbol{G} \cong \bigoplus_\lambda \mathbb{Z}$.
- $\boldsymbol{G}$ is κ-**free** iff every subgroup of $\boldsymbol{G}$ of rank $< \kappa$ is free.

Fact

If $\boldsymbol{G}$ is κ-free, for some infinite κ, then $\boldsymbol{G}$ is **torsion-free**, *i.e., every nontrivial element of $\boldsymbol{G}$ has infinite order.*

Compactness in the universe

Theorem (Pontryagin 1934)

Every countable $\aleph_0$-free group is free.

Theorem (Folklore)

If κ is weakly compact, then every κ-free group of cardinality κ is free.

Theorem (Shelah 1975)

If κ is singular, then every κ-free group of cardinality κ is free.

A long list of other applications

Other remarkable applications of pure set theory to abelian groups include:

Undecidability of Whitehead's problems. (Shelah)

When κ-free implies κ^+-free. (Magidor, Shelah)

Consequences of PFA on the classification of $\aleph_1$-separable abelian groups. (Eklof)

$\ldots$

How about descriptive set theory?

The last two decades have seen an increasing interest in TFA groups by descriptive set theorists.

Some natural equivalence relations on TFA groups can serve as **milestones** in the hierarchy of **analytic equivalence relations**.

Borel classification

Definition

Suppose that $(\mathcal{X}, \cong_{\mathcal{X}})$ and $(\mathcal{Y}, \cong_{\mathcal{Y}})$ are two standard Borel spaces with two corresponding equivalence relations. We say that $\cong_{\mathcal{X}}$ is **Borel reducible to** $\cong_{\mathcal{Y}}$ iff there exists a Borel $\phi\colon \mathcal{X} \to \mathcal{Y}$ such that

$$x \cong_{\mathcal{X}} x' \quad \iff \quad \phi(x) \cong_{\mathcal{Y}} \phi(x').$$

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We can view Borel reducibility in two ways.

- $\cong_{\mathcal{Y}}$-classes are complete invariants for $\cong_{\mathcal{X}}$ (Borel complexity).
- There is an injection of $\mathcal{X}/{\cong_{\mathcal{X}}}$ into $\mathcal{Y}/{\cong_{\mathcal{Y}}}$ admitting Borel lifting (Borel cardinality).

Why do we bother?

We can form **standard Borel spaces** of well-known **mathematical structures** (e.g., $\mathcal{L}_{\omega_1\omega}$-elementary class of countable structures, separable Banach spaces, ...), and then

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- perform a fine **analysis of suitable invariants** (reals, countable sets of reals, orbits of group actions, ...);
- find strong **evidence against classification** (Borel/not Borel, turbulence, ...);
- in a single catch phrase by E.G. Effros:
  "Classifying the unclassifiables".

The space of countable almost-free groups

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- Let X_{TFA} be the set of all torsion-free abelian groups on $\mathbb{N}$.
- Each group G is identified with a function $m_G \in 2^{\mathbb{N}^3}$ by setting

$$m_G(a, b, c) \iff a +_G b = c, \quad \text{for } a, b, c \in \mathbb{N}.$$

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- $X_{\mathsf{TFA}} \subseteq 2^{\mathbb{N}^3}$ is Borel (and closed under isomorphism) so it is standard Borel[1].

[1] In fact, X_{TFA} with the induced topology form a Polish space, since it is G_δ.

A long-standing conjecture

Conjecture (Friedman-Stanley 1989)
Every isomorphism relation $\cong$ *is Borel reducible to isomorphism* $\cong_{\mathsf{TFA}}$ *on torsion-free abelian groups.*

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Theorem (Hjorth 2002)

$\cong_{\mathsf{TFA}}$ *is not Borel.*

Theorem (Downey-Montalban 2008)

$\cong_{\mathsf{TFA}}$ *is complete* $\mathbf{\Sigma}^1_1$ *as a subset of* $X_{\mathsf{TFA}} \times X_{\mathsf{TFA}}$.

Theorem (Shelah-Ulrich)

It is consistent with ZFC that every isomorphism is a$\mathbf{\Delta}^1_2$*-reducible to* $\cong_{\mathsf{TFA}}$.

Definition

$A \sqsubseteq_{\mathsf{TFA}} B$ iff there exists an injective homomorphism $h\colon A \to B$.

$A \equiv_{\mathsf{TFA}} B$ iff $A \sqsubseteq_{\mathsf{TFA}} B$ and $B \sqsubseteq_{\mathsf{TFA}} A$.

Bi-embeddability on TFA groups

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Theorem (C.-Thomas 2019)

Every $\boldsymbol{\Sigma}^1_1$ equivalence relation is Borel reducible to the bi-embeddability relation $\equiv_{\mathsf{TFA}}$ on torsion-free abelian group. Thus it is strictly more complicated than $\cong_{\mathsf{TFA}}$.

Higher descriptive set theory

Many people have developed the generalized version of Borel classification for higher structures (Friedman, Hyttinen, Kulikov, Moreno, Motto Ros, . . .).

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Let κ be uncountable such that $\kappa = \kappa^{<\kappa}$.

Theorem (C. 2018)

Every $\mathbf{\Sigma}^1_1$ equivalence relation on a standard Borel κ-space is Borel reducible to the bi-embeddability relation $\equiv^{\kappa}_{\mathsf{TFA}}$ on κ-sized torsion-free abelian group.

- Obtained before C.-Thomas.
- Proofs are very mach different.

Ordered TFA groups. Prelude

Let $\cong_{\mathsf{DAG}}$ be the isomorphism relation of torsion-free divisible abelian groups.

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Fact

A torsion-free abelian group is **divisible** *if and only if*

$$A = \underbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}_{rk(A)}.$$

We have $A \cong B$ iff $rk(A) = rk(B)$. Thus, $\cong_{\mathsf{DAG}}$ is Borel reducible to $=_{\mathbb{N}}$.

Ordered TFA groups. Act I

Let $\cong_{\mathrm{ODAG}}$ be the (increasing) isomorphism relation on ordered divisible abelian groups.

A group $(G, +, <)$ is **ordered** if $<$ is a linear order on G and

$$x < y \implies x + z < y + z.$$

An ordered group is necessarily torsion-free.

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Fact

Every isomorphism relation is Borel reducible to $\cong_{\text{ODAG}}$.

Theorem (Rast-Sahota 2016)

If T is an o-minimal theory and has a nonsimple type, then $\cong_{\text{LO}}$ is Borel reducible to isomorphism $\cong_T$ on countable models of T.

Ordered TFA groups. I act

Theorem (C.-Marker-Motto Ros)
Every $\mathbf{\Sigma}^1_1$ *equivalence relation is Borel reducible to* $\equiv_{\mathsf{ODAG}}$.

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Cannot use linear orders but . . . we can color them!

Definition

A **colored linear order** on $\mathbb{N}$ is a pair $L = (<_L, c_L)$ such that $<_L$ is a linear order on $\mathbb{N}$ and $c_L \colon \mathbb{N} \to \mathbb{N}$. All CLOs on $\mathbb{N}$ form a Polish space.

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$K \sqsubseteq_{\mathsf{CLO}} L$ if and only if there exists $f \colon \mathbb{N} \to \mathbb{N}$ such that

- $m <_K n$ implies $f(m) <_L f(n)$ for every $m, n \in \mathbb{N}$;
- $c_L(f(n)) = c_K(n)$ for every $n \in \mathbb{N}$.

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Theorem (Louveau, Marcone-Rosendal 2004)

Every $\boldsymbol{\Sigma}^1_1$ equivalence relation is Borel reducible to $\equiv_{\mathsf{CLO}}$.

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We produce a reduction from $\equiv_{\mathrm{CLO}}$ to $\equiv_{\mathrm{ODAG}}$.

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Given $L = (<_L, c_L)$ we define G_L as the group of finite support functions

$$f\colon L \to \bigsqcup H_n \quad \text{s.t.} \quad f(n) \in H_k \iff c_L(n) = k.$$

We order G_L antilexicographically.

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When we color $\ell(G_L)$ in the obvious way, $\ell(G_L) \cong_{\mathsf{CLO}} L$. $\square$

Ordered TFA groups. II act

Consider the (increasing) isomorphism relation $\cong_{\mathsf{ArGP}}$ on countable Archimedean groups.

Theorem (Hölder)

An ordered group is Archimedean iff it is a subgroup of $(\mathbb{R}, +)$*. Thus, every Archimedean group is Abelian.*

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Fact

$\phi\colon A \to B$ *is an increasing homomorphism iff there exists* $r \in \mathbb{R}^+$ *such that* $\phi(a) = r \cdot a$*.*

It follows that $\cong_{\mathsf{ArGp}}$ is Borel.

Potential complexity

Definition

Let E be a Borel equivalence relation on X. The equivalence relation E^+ on $X^{\mathbb{N}}$ is defined by

$$(x_n)\, E^+\, (y_n) \quad \iff \quad \{[x_n]_E : n \in \mathbb{N}\} = \{[y_n]_E : n \in \mathbb{N}\}.$$

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Definition (Hjorth-Kechris-Louveau)

Suppose that E is an equivalence relation on a standard Borel space X. We say that E is **potentially in** Γ if there exists a Polish topology τ generating the Borel structure of X such that E is in Γ in the product space $(X \times X, \tau^2)$.

An upper bound

Theorem (Hjorth-Kechris-Louveau)
Suppose that E is a Borel equivalence relation on a standard Borel space, and E is induced by a Borel action of a closed subgroup G of S_∞. Then E is potentially $\mathbf{\Pi}^0_3$ iff E is Borel reducible to $(=_\mathbb{R})^+$.

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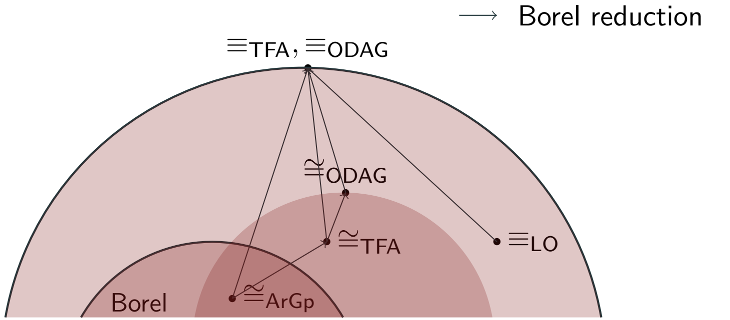
Theorem (C.-Marker-Motto Ros)

$\cong_{\mathsf{ArGp}}$ is potentially $\boldsymbol{\Sigma}^0_4$. Thus, $\cong_{\mathsf{ArGp}}$ is Borel reducible $(=_{\mathbb{R}})^{+++}$.

On the other hand, $=^+_{\mathbb{R}}$ is Borel reducible to $\cong_{\mathsf{ArGp}}$.

Analytic equivalence relations



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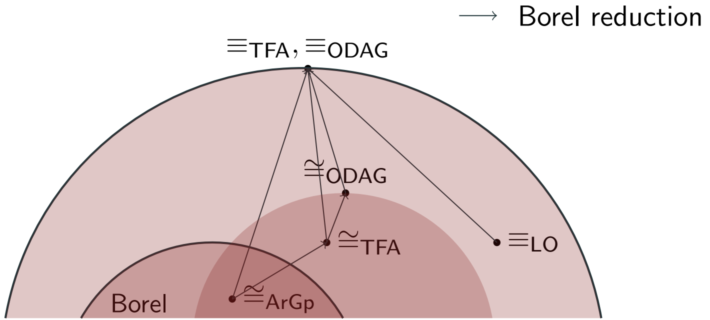

Thank you!