Side conditions and revisionism

David Asperó

University of East Anglia

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Forcing with symmetric systems of models as side conditions

Finite–support forcing iterations involving symmetric systems of models as side conditions are useful in situations in which, for example, we want to force

- consequences of classical forcing axioms at the level of $H(\omega_2)$, together with
- $2^{\aleph_0}$ large.

Given a cardinal κ and $T \subseteq H(\kappa)$, a finite $\mathcal{N} \subseteq [H(\kappa)]^{\aleph_0}$ is a *T–symmetric system* if

(1) for every $N \in \mathcal{N}$,

$$(N, \in, T) \preccurlyeq (H(\kappa), \in, T),$$

(2) given N_0, $N_1 \in \mathcal{N}$, if $N_0 \cap \omega_1 = N_1 \cap \omega_1$, then there is a unique isomorphism

$$\Psi_{N_0,N_1} : (N_0, \in, T) \longrightarrow (N_1, \in, T)$$

and Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$.

(3) Given N_0, $N_1 \in \mathcal{N}$ such that $N_0 \cap \omega_1 = N_1 \cap \omega_1$ and $M \in N_0 \cap \mathcal{N}$, $\Psi_{N_0,N_1}(M) \in \mathcal{N}$.

(4) Given M, $N_0 \in \mathcal{N}$ such that $M \cap \omega_1 < N_0 \cap \omega_1$, there is some $N_1 \in \mathcal{N}$ such that $N_1 \cap \omega_1 = N_0 \cap \omega_1$ and $M \in N_1$.

The pure side condition forcing

$$\mathcal{P}_0 = (\{\mathcal{N} : \mathcal{N} \text{ a } T\text{–symmetric system}\}, \supseteq)$$

(for any fixed $T \subseteq H(\kappa)$) preserves CH:

This exploits the fact that given N, $N' \in \mathcal{N}$, $\mathcal{N}$ a symmetric system, if $N \cap \omega_1 = N' \cap \omega_1$, then $\Psi_{N,N'}$ is an isomorphism

$$\Psi_{N,N'} : (N; \in, \mathcal{N} \cap N) \longrightarrow (N'; \in, \mathcal{N} \cap N')$$

Proof: Suppose $(\dot{r}_\xi)_{\xi<\omega_2}$ are names for subsets of ω and $\mathcal{N} \Vdash_{\mathcal{P}_0} \dot{r}_\xi \neq \dot{r}_{\xi'}$ for all $\xi \neq \xi'$. For each ξ, let N_ξ be a sufficiently correct model such that $\mathcal{N}$, $\dot{r}_\xi \in N_\xi$.

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By CH we may find $\xi \neq \xi'$ such that there is an isomorphism

$$\Psi : (N_\xi; \in, T^*, \mathcal{N}, \dot{r}_\xi) \longrightarrow (N_{\xi'}; \in, T^*, \mathcal{N}, \dot{r}_{\xi'})$$

(where T^* is the satisfaction predicate for $(H(\kappa); \in, T)$). Then $\mathcal{N}^* = \mathcal{N} \cup \{N_\xi, N_{\xi'}\} \in \mathcal{P}_0$. But $\mathcal{N}^*$ is $(N_\xi, \mathcal{P}_0)$–generic and $(N_{\xi'}, \mathcal{P}_0)$–generic.

Now, let $n < \omega$ and let $\mathcal{N}'$ be an extension of $\mathcal{N}^*$. Suppose $\mathcal{N}' \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. Then there is $\mathcal{N}'' \in \mathcal{P}_0$ extending both $\mathcal{N}'$ and some $\mathcal{M} \in N_\xi \cap \mathcal{P}_0$ such that $\mathcal{M} \Vdash_{\mathcal{P}_0} n \in \dot{r}_\xi$. **By symmetry,** $\mathcal{N}''$ extends also $\Psi(\mathcal{M})$. But $\Psi(\mathcal{M}) \Vdash_{\mathcal{P}_0} n \in \Psi(\dot{r}_\xi) = \dot{r}_{\xi'}$.

We have shown $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_\xi \subseteq \dot{r}_{\xi'}$, and similarly we can show $\mathcal{N}^* \Vdash_{\mathcal{P}_0} \dot{r}_{\xi'} \subseteq \dot{r}_\xi$. Contradiction since $\mathcal{N}^*$ extends $\mathcal{N}$ and $\xi \neq \xi'$. $\square$

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$\square$

In typical forcing iterations with symmetric systems as side conditions, $2^{\aleph_0}$ is large in the final extension. Even if $\mathcal{P}_0$ can be seen as the first stage of these iterations, the forcing is in fact designed to add reals at (all) subsequent successor stages.

Something one may want to try at this point: Extend the symmetry requirements **also** to the working parts in such a way that the above CH–preservation argument goes trough. Hope to be able to force something interesting this way.

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A toy example: Getting a model of the negation of Weak Club Guessing with CH

Weak Club Guessing (WCG): For every ladder system $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ (i.e., each C_δ is a cofinal subset of δ of order type ω) there is a club $C \subseteq \omega_1$ such that $C \cap C_\delta$ is finite for all δ.

(Shelah, NNR revisited): $\neg$ WCG is consistent with CH.

As with many classical results in the area this is done by building a countable–support iteration dealing with the relevant problem. At successor stages no new reals are added. The bulk of the proof is by far in showing that no new reals are added at limit stages either.

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The following is an outline of a proof of this result using side conditions **and adding reals.**

We start with GCH. Fix $\Phi : \omega_2 \to H(\omega_2)$ such that $\Phi^{-1}(x)$ is unbounded in ω_2 for all $x \in H(\omega_2)$. We build $(\mathcal{P}_\beta \, : \, \beta < \omega_2)$:

Given β such that $\mathcal{P}_\alpha$ has been defined for all $\alpha < \beta$, we define $\mathcal{P}_\beta$.

$q = (F, \Delta, \tau)$ is a condition in $\mathcal{P}_\beta$ iff:

(1) Δ is a finite collection of pairs (N, γ) such that N is an elementary submodel of $H(\omega_2)$, $\gamma \leq \beta$, and γ is in the closure of $N \cap \mathrm{Ord}$.

(2) $\mathrm{dom}(\Delta)$ is a symmetric system of countable elementary submodels of $H(\omega_2)$.

(3) F is a finite function with $\mathrm{dom}(F) \subseteq \beta$.

(4) For every $\alpha \in \mathrm{dom}(F)$, if $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$-name for a ladder system $\vec{C}_\alpha = (C^\alpha_\delta : \delta \in \mathrm{Lim}(\omega_1))$, then $F(\alpha)$ is a condition for a natural forcing $\mathcal{Q}_{\vec{C}_\alpha}$ for adding a club of ω_1, via finite collections of disjoint intervals, with finite intersection with C^α_δ for each δ.

(5) For every $(N, \gamma) \in \Delta$ and $\alpha \in \mathrm{dom}(F)$, if $\alpha \in N \cap \gamma$, then $\delta_N := N \cap \omega_1$ is in the club added at stage α.

(6) τ is a collection of pairs $((N_0, \gamma_0), (N_1, \gamma_1))$ such that N_0, $N_1 \in \mathrm{dom}(\Delta)$, $\delta_{N_0} = \delta_{N_1}$, and γ_0, $\gamma_1 \leq \beta$ are in the closure of $N_0 \cap \mathrm{Ord}$ and $N_1 \cap \mathrm{Ord}$, resp. Members of τ are called **edges.**

(7) $q|_\alpha := (F \restriction \alpha, \Delta \restriction \alpha, \tau \restriction \alpha) \in \mathcal{P}_\alpha$ for all $\alpha < \beta$.

Main ingredient: Revisionism (copying information from the future into the past).

(8) Given $((N_0, \gamma_0), (N_1, \gamma_1)) \in \tau$, $\Psi_{N_0,N_1}(\xi) \leq \xi$ for every ordinal $\xi \in N_0$ (so N_1 is a 'projection of N_0').

(9) Given $((N_0, \gamma_0), (N_1, \gamma_1)) \in \tau$ and $\alpha \in N_0 \cap \gamma_0$ such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$, **all information carried by the condition at α inside N_0 is copied on $\Psi_{N_0,N_1}(\alpha)$.**

Given $\mathcal{P}_\beta$-conditions $q_0 = (F_0, \Delta_0, \tau_0)$, $q_1 = (F_1, \Delta_1, \tau_1)$, $q_1 \leq_\beta q_0$ iff

- $\Delta_0 \subseteq \Delta_1$,
- $\tau_0 \subseteq \tau_1$,
- $\text{dom}(F_0) \subseteq \text{dom}(F_1)$, and
- for each $\alpha \in \text{dom}(F_0)$,

$$q_1|_\alpha \Vdash_{\mathcal{P}_\alpha} F_1(\alpha) \leq_{\mathcal{Q}_{\vec{C}_\alpha}} F_0(\alpha)$$

Finally, $\mathcal{P}_{\omega_2} = \bigcup_{\beta<\omega_2} \mathcal{P}_\beta$.

Main facts

(0) Thanks to the fact that we are only copying information 'from the future into the past', $(\mathcal{P}_\beta)_{\beta \leq \omega_2}$ is a forcing iteration (i.e., $\mathcal{P}_\alpha \lessdot \mathcal{P}_\beta$ for all $\alpha < \beta$): Given $q \in \mathcal{P}_\beta$ and $r \in \mathcal{P}_\alpha$, if $r \leq_\alpha q \restriction \alpha$, then

$$(F_r \cup F_q \restriction [\alpha, \beta), \Delta_q \cup \Delta_r, \tau_q \cup \tau_r)$$

is a common extension of q and r in $\mathcal{P}_\beta$.

(1) $\mathcal{P}_{\omega_2}$ has the $\aleph_2$-c.c. [thanks to CH, by standard Δ-system argument].

(2) $\mathcal{P}_\beta$ is proper for all $\beta \leq \omega_2$ [natural proof by induction on β, using finiteness of supports and the basic structural properties of symmetric systems].

(3) $\mathcal{P}_{\omega_2}$ adds $\aleph_1$-many new reals (in fact Cohen reals), but not more than that; in particular, $\mathcal{P}_{\omega_2}$ preserves CH [essentially the same argument we saw a few slides back].

(4) $\mathcal{P}_{\omega_2}$ forces $\neg$ WCG [standard density argument, since $\mathcal{P}_{\omega_2}$ is $\aleph_2$-c.c.]

A pretty optimal form of this construction

Measuring is the following very strong form of $\neg$ WCG: Let $(C_\delta \, : \, \delta \in \mathrm{Lim}(\omega_1))$ such that for all α, C_δ is a closed subset of δ with the order topology. Then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$, a either

- a tail of $C \cap \delta$ is contained in C_δ , or
- a tail of $C \cap \delta$ is disjoint from C_δ.

Question: (J. Moore) Is Measuring compatible with CH?

In joint work with M.A. Mota, we answered this question affirmatively using variation of above construction for $\neg$ WCG+CH.

The following question addresses the issue whether adding new reals is a necessary feature of any approach to forcing Measuring.

Question: (J. Moore) Does Measuring imply the existence of a non–constructible real?

Let's get high.

$\aleph_2$-Suslin trees

Jensen (1972) proved that the existence of an $\aleph_2$-Suslin tree follows from each of the hypotheses
$\mathsf{CH} + \diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega_1\})$ and
$\square_{\omega_1} + \diamondsuit(\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\})$.

Gregory (1976) proved that GCH together with the existence of a non–reflecting stationary subset of $\{\alpha < \omega_2 \mid \mathrm{cf}(\alpha) = \omega\}$ yields the existence of an $\aleph_2$-Suslin tree.

Theorem

(Laver–Shelah, 1981) If there is a weakly compact cardinal κ, then there is a forcing extension in which $\kappa = \aleph_2$, CH holds, and all $\aleph_2$-Aronszajn trees are special (and hence there are no $\aleph_2$-Suslin trees).

The proof proceeds by

- Lévy–collapsing κ to become ω_2, and then
- running a countable–support iteration of length κ^+ in which one specializes, with countable conditions, all κ-Aronszajn trees given by some book-keeping function.
- One uses the weak compactness of κ in V in a crucial way in order to show that the iteration has the κ-c.c. and hence everything goes as planned.

In the Laver–Shelah model, $2^{\aleph_1} = \aleph_3$, and the following remained a major open problem (s. e.g. Kanamori–Magidor 1977):

Question

Is ZFC+GCH consistent with the non–existence of $\aleph_2$-Suslin trees?

At least a weakly compact cardinal is needed for a Yes answer:

(1) (Rinot) If GCH holds, $\lambda \geq \omega_1$ is a cardinal, and $\square(\lambda^+)$ holds, then there is a λ-closed λ^+-Suslin tree.

(2) (Todorčević) If $\kappa \geq \omega_2$ is regular and $\square(\kappa)$ fails, then κ is weakly compact in L.

While Visiting Mohammad Golshani in Tehran in December 2017, we thought about applying the ideas for preserving CH with side conditions (with $2^{\aleph_1} = \aleph_2$ instead of $2^{\aleph_0} = \aleph_1$ and $\aleph_1$-sized models instead if countable models) to the Laver–Shelah construction, in order to build a model of GCH with no $\aleph_2$-Suslin trees. We eventually succeeded:

The result

Theorem (A.–Golshani) Suppose κ is a weakly compact cardinal. Then there exists a generic extension of the universe in which

1. GCH holds,
2. $\kappa = \aleph_2$, and
3. all $\aleph_2$-Aronszajn trees are special (and hence there are no $\aleph_2$-Suslin trees).

Remark

The same proof works replacing ω_2 with λ^+ for any regular $\lambda \geq \omega_1$.

Proof sketch

Let κ be weakly compact. W.l.o.g. we may assume $2^\mu = \mu^+$ for all $\mu \geq \kappa$.
Let

$$\Phi : \kappa^+ \to H(\kappa^+)$$

be such that for each $x \in H(\kappa^+)$, $\Phi^{-1}(x)$ is an unbounded subset of κ^+. Φ exists by $2^\kappa = \kappa^+$.

Let also $(\Phi_\alpha)_{\alpha<\kappa^+}$ be a sequence of increasingly expressive (satisfaction) predicates of $H(\kappa^+)$ such that $\Phi_0 = \Phi$.

Let us call

$$\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$$

an *edge below* β if

(0) For all $i \in \{0, 1\}$, $N_i \subseteq H(\kappa^+)$, $\delta_{N_i} := N \cap \kappa \in \kappa$, $|N_i| = |\delta_{N_i}|$, and ${}^{<|N_i|}N_i \subseteq N_i$.

(1) For all $i \in \{0, 1\}$, γ_i is an ordinal in the closure of $N_i \cap \{\xi + 1 \,:\, \xi < \beta\}$ and $(N_i, \in, \Phi_\alpha) \preccurlyeq (H(\kappa^+), \in, \Phi_\alpha)$ for all $\alpha \in N_i \cap \gamma_i$.

(2) $N_0 \cong N_1$ via an isomorphism $\Psi_{N_0, N_1} : N_0 \longrightarrow N_1$ such that
   (i) $(N_0, \in, \Phi_\alpha) \cong (N_1, \in, \Phi_{\Psi_{N_0,N_1}(\alpha)})$ for all $\alpha < \gamma_0$ such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$,
   (ii) Ψ_{N_0,N_1} is the identity on $N_0 \cap N_1$, and
   (iii) $\Psi_{N_0,N_1}(\xi) \leq \xi$ for every ordinal $\xi \in N_0$.

Given $\beta \leq \kappa^+$, we will build $\mathbb{Q}_\beta$ as a forcing with side conditions consisting of sets of edges below β.

Given an edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ in the side condition, we will copy information in N_0 attached to $\alpha < \gamma_0$ via Ψ_{N_0,N_1} into N_1 if $\Psi_{N_0,N_1}(\alpha) < \gamma_1$.

We do not require that information in N_1 attached to $\Psi_{N_0,N_1}(\alpha)$ be copied into N_0.

Definition of the forcing

Let $\beta \leq \kappa^+$ and suppose $\mathbb{Q}_\alpha$ defined for all $\alpha < \beta$. A condition in $\mathbb{Q}_\beta$ is an ordered pair of the form $q = (f_q, \tau_q)$ with the following properties.

(1) f_q is a countable function such that $\mathrm{dom}(f_q) \subseteq \kappa^+ \cap \beta$ and such that the following holds for every $\alpha \in \mathrm{dom}(f_q)$.

  (a) If $\alpha = 0$, then $f_q(\alpha) \in \mathrm{Col}(\omega_1, {<}\kappa)$.

  (b) If $\alpha > 0$, then

$$f_q(\alpha) : \kappa \times \omega_1 \to \omega_1$$

  is a countable function.

(2) τ_q is a countable set of edges below β.

(3) For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$, if $\alpha \in N_0 \cap \gamma_0$ is such that $\Psi_{N_0,N_1}(\alpha) < \gamma_1$, then *every piece of information about* q *at* α *inside* N_0 *is to be copied at* $\Psi_{N_0,N_1}(\alpha)$ *via* Ψ_{N_0,N_1}.

(4) For all $\alpha < \beta$, $q \restriction \alpha \in \mathbb{Q}_\alpha$, where

$$q \restriction \alpha = (f_q \restriction \alpha, \tau_q \restriction \alpha)$$

(5) The following holds for every nonzero $\alpha < \beta$.

(a) If $\alpha \in \mathrm{dom}(f_q)$, then $q \upharpoonright \alpha$ forces that $f_q(\alpha)$ is a partial specializing function for $\underset{\sim}{T}_\alpha$.

(b) For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$, if $\alpha \in N_0 \cap \gamma_0$, then $\mathbb{Q}_{\alpha+1} \cap N_0 \lessdot \mathbb{Q}^{N_0}_{\alpha+1}$, where $\mathbb{Q}^{N_0}_{\alpha+1}$ is the partial order whose conditions are ordered pairs $p = (f_p, \tau_p)$ such that

(i) f_p is a function such that $\mathrm{dom}(f_p) \subseteq \alpha + 1$,

(ii) if $\alpha \in \mathrm{dom}(f_p)$, then $f_p(\alpha) : \kappa \times \omega_1 \to \omega_1$ is a countable function,

(iii) τ_p is a set of edges below $\alpha + 1$,

(iv) $\gamma_0, \gamma_1 \leq \alpha$ for every $\langle (N_0', \gamma_0), (N_1', \gamma_1) \rangle \in \tau_p \setminus N_0$,

(v) $p \upharpoonright \alpha \in \mathbb{Q}_\alpha$,

(vi) $p \upharpoonright N_0 \in \mathbb{Q}_{\alpha+1}$, and

(vii) if $\alpha \in \mathrm{dom}(f_p)$, then $p \upharpoonright \alpha$ forces that $f_p(\alpha)$ is a partial specializing function for $\underset{\sim}{T}_\alpha$,

ordered by setting $p_1 \leq_{\mathbb{Q}^{N_0}_{\alpha+1}} p_0$ if

- $p_1 \upharpoonright \alpha \leq_{\mathbb{Q}_\alpha} p_0 \upharpoonright \alpha$ and
- $f_{p_0}(\alpha) \subseteq f_{p_1}(\alpha)$ in case $\alpha \in \mathrm{dom}(f_{p_0})$.

The extension relation:

Given $q_1, q_0 \in \mathbb{Q}_\beta$, $q_1 \leq_\beta q_0$ (q_1 is an extension of q_0) if and only if the following holds.

(A) $\mathrm{dom}(f_{q_0}) \subseteq \mathrm{dom}(f_{q_1})$

(B) for every $\alpha \in \mathrm{dom}(f_{q_0})$, $f_{q_0}(\alpha) \subseteq f_{q_1}(\alpha)$.

(C) For every $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_{q_0}$ there are $\gamma_0' \geq \gamma_0$ and $\gamma_1' \geq \gamma_1$ such that $\langle (N_0, \gamma_0'), (N_1, \gamma_1') \rangle \in \tau_{q_1}$.

Main facts

(0) For every $\beta < \kappa^+$, $\mathbb{Q}_\beta$ is definable over the structure

$$(H(\kappa^+), \in, \Phi_{\beta+1})$$

without parameters. Moreover, this definition can be taken to be uniform in β.

(1) $\mathbb{Q}_1$ forces $\kappa = \omega_2$.

(2) For every $\beta \leq \kappa^+$,

  (i) $\mathbb{Q}_\alpha \subseteq \mathbb{Q}_\beta$ for all $\alpha < \beta$, and

  (ii) if $\mathrm{cf}(\beta) \geq \kappa$, then $\mathbb{Q}_\beta = \bigcup_{\alpha<\beta} \mathbb{Q}_\alpha$.

(3) Thanks to the fact that we are only copying information ‘from the future into the past’, $(\mathbb{Q}_\beta)_{\beta \leq \kappa^+}$ is a forcing iteration (i.e., $\mathbb{Q}_\alpha \lessdot \mathbb{Q}_\beta$ for all $\alpha < \beta$): Given $q \in \mathbb{Q}_\beta$ and $r \in \mathbb{Q}_\alpha$, if $r \leq_\alpha q \restriction \alpha$, then

$$(f_r \cup f_q \restriction [\alpha, \beta), \tau_q \cup \tau_r)$$

is a common extension of q and r in $\mathbb{Q}_\beta$.

(4) $\mathbb{Q}_\beta$ is σ-closed for every $\beta \leq \kappa^+$. In fact, every decreasing ω-sequence $(f_n)_{n<\omega}$ of $\mathbb{Q}_\beta$-conditions has a greatest lower bound q^* in $\mathbb{Q}_\beta$, $q^* = (f, \bigcup_n \tau_{q_n})$, where $\mathrm{dom}(f) = \bigcup_n \mathrm{dom}(f_{q_n})$, and

$$f(\alpha) = \bigcup\{f_{q_m}(\alpha) \,:\, m \geq n\}$$

for all n and $\alpha \in \mathrm{dom}(f_{q_n})$. In particular, forcing with $\mathbb{Q}_\beta$ does not add new ω-sequences of ordinals, and therefore it preserves both ω_1 and CH.

(5) If $\mathbb{Q}_{\kappa^+}$ has the κ-c.c., then it adds κ-many new subsets of ω_1, but not more than that; in particular, $\mathbb{Q}_{\kappa^+}$ preserves $2^{\aleph_1} = \aleph_2$ [essentially the same argument we saw a few slides back].

(6) If $\mathbb{Q}_{\kappa^+}$ has the κ-c.c., then it forces that all $\aleph_2$-Aronszajn are special.

(7) For each $\beta \leq \kappa^+$, $\mathbb{Q}_\beta$ has the κ-c.c.

No symmetric systems are needed in the construction thanks to the fact that the $\mathbb{Q}_\beta$'s are, not only proper for suitable κ-sized models N, but in fact have the κ-c.c. (so $A \subseteq N$ whenever $A \in N$ is a maximal antichain).

The κ–chain condition: Proof sketch

We call a model Q *suitable* if Q is an elementary submodel of cardinality κ of some high enough $H(\theta)$, closed under $<\kappa$-sequences, and such that $\langle \mathbb{Q}_\alpha \mid \alpha < \kappa^+ \rangle \in Q$. Given a suitable model Q, a bijection $\varphi : \kappa \to Q$, and an ordinal $\lambda < \kappa$, we will denote $\varphi``\lambda$ by M^φ_λ.

Let $\mathcal{F}$ be the weak compactness filter on κ, i.e., the filter on κ generated by the sets

$$\{\lambda < \kappa \mid (V_\lambda, \in, B \cap V_\lambda) \models \psi\},$$

where $B \subseteq V_\kappa$ and where ψ is a Π^1_1 sentence for the structure $(V_\kappa, \in, B)$ such that

$$(V_\kappa, \in, B) \models \psi$$

$\mathcal{F}$ is a proper normal filter on κ. Let also $\mathcal{S}$ be the collection of $\mathcal{F}$-positive subsets of κ, i.e.,

$$\mathcal{S} = \{X \subseteq \kappa \mid X \cap C \neq \emptyset \text{ for all } C \in \mathcal{F}\}$$

Given $\beta \leq \kappa^+$, we will say that $\mathbb{Q}_\beta$ *has the strong κ-chain condition* if for every $X \in \mathcal{S}$, every suitable model Q such that $\beta, X \in Q$, every bijection $\varphi : \kappa \to Q$, and every two sequences

$$(q^0_\lambda \mid \lambda \in X) \in Q$$

and

$$(q^1_\lambda \mid \lambda \in X) \in Q$$

of $\mathbb{Q}_\beta$-conditions, if $q^0_\lambda \restriction M^\varphi_\lambda = q^1_\lambda \restriction M^\varphi_\lambda$ for every $\lambda \in X$, then there is some $Y \in \mathcal{S}$, $Y \subseteq X$, together with sequences

$$(q^{00}_\lambda \mid \lambda \in Y)$$

and

$$(q^{11}_\lambda \mid \lambda \in Y)$$

of $\mathbb{Q}_\beta$-conditions with the following properties.

(1) $q^{00}_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $q^{11}_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ for every $\lambda \in Y$.

(2) For all $\lambda < \lambda^*$ in Y, $q^{00}_\lambda \oplus q^{11}_{\lambda^*}$ is a common extension of q^{00}_λ and $q^{11}_{\lambda^*}$.

Given a suitable model Q such that $\beta \in Q$, a bijection $\varphi : \kappa \to Q$, a $\mathbb{Q}_\beta$-condition $q \in Q$, and $\lambda < \kappa$, let us say that q *is λ-compatible with respect to φ and β* if, letting $\mathbb{Q}^*_\beta = \mathbb{Q}_\beta \cap Q$, we have that

- $\mathbb{Q}^*_\beta \cap M^\varphi_\lambda \lessdot \mathbb{Q}^*_\beta$,
- $q \restriction M^\varphi_\lambda \in \mathbb{Q}^*_\beta$, and
- $q \restriction M^\varphi_\lambda$ forces in $\mathbb{Q}^*_\beta \cap M^\varphi_\lambda$ that q is in the quotient forcing $\mathbb{Q}^*_\beta / \dot{G}_{\mathbb{Q}^*_\beta \cap M^\varphi_\lambda}$; equivalently, for every $r \leq_{\mathbb{Q}^*_\beta \cap M^\varphi_\lambda} q \restriction M^\varphi_\lambda$, r is compatible with q.

Given $\alpha < \kappa^+$ and given nodes x, $y \in \kappa \times \omega_1$, if $\mathbb{Q}_\alpha$ is κ-c.c., then we denote by $A^\alpha_{x,y}$ the first, in some well–order of $H(\kappa^+)$ canonically definable from Φ, maximal antichain of $\mathbb{Q}_\alpha$ consisting of conditions deciding whether or not x and y are comparable in $\utilde{T}_\alpha$.

Given $q \in \mathbb{Q}_{\kappa^+}$, we will say that q is *adequate* in case:

(1) For all nonzero α, α' in $\mathrm{dom}(f_q)$, if $x \in \mathrm{dom}(f_q(\alpha))$, $y \in \mathrm{dom}(f_q(\alpha'))$, and $\mathbb{Q}_\alpha$ is κ-c.c., then $q \restriction \alpha$ extends a condition in $A^\alpha_{x,y}$.

(2) For every edge $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_q$ and every $\alpha \in \mathrm{dom}(f_q) \cap N_1 \cap \gamma_1$, if $\Psi_{N_1,N_0}(\alpha) < \gamma_0$, then $\Psi_{N_1,N_0}(\alpha) \in \mathrm{dom}(f_q)$ and

$$f_q(\Psi_{N_1,N_0}(\alpha)) \restriction \delta_{N_1} \times \omega_1 = f_q(\alpha) \restriction \delta_{N_1} \times \omega_1$$

Lemma

For every $\beta \le \kappa^+$, the set of adequate $\mathbb{Q}_\beta$-conditions is dense in $\mathbb{Q}_\beta$.

Rather than proving that every $\mathbb{Q}_\beta$ has the κ-c.c., we prove the following more informative lemma by induction on β.

Lemma

The following holds for every $\beta \leq \kappa^+$.

$(1)_\beta$ $\mathbb{Q}_\beta$ *has the strong κ-chain condition.*

$(2)_\beta$ *Suppose $D \in \mathcal{F}$, Q is a suitable model, β, $D \in Q$, $\varphi : \kappa \to Q$ is a bijection, and $(q^0_\lambda \mid \lambda \in D) \in Q$ and $(q^1_\lambda \mid \lambda \in D) \in Q$ are sequences of adequate $\mathbb{Q}_\beta$-conditions. Then there is some $D' \in \mathcal{F}$ such that $D' \subseteq D$ and such that for every $\lambda \in D'$, if $q^0_\lambda \restriction M^\varphi_\lambda = q^1_\lambda \restriction M^\varphi_\lambda$, then there are conditions $q'^0_\lambda \leq_{\mathbb{Q}_\beta} q^0_\lambda$ and $q'^1_\lambda \leq_{\mathbb{Q}_\beta} q^1_\lambda$ such that*

(a) $q'^0_\lambda \restriction M^\varphi_\lambda = q'^1_\lambda \restriction M^\varphi_\lambda$ *and*

(b) q'^0_λ *and* q'^1_λ *are both λ-compatible with respect to φ and β.*

The proof of the lemma is an adaptation of the Laver–Shelah argument for proving κ-c.c. of their forcing.

An open question

Question (Shelah): Is it consistent to have GCH together with a successor cardinal $\kappa \geq \omega_1$ such that all κ-Aronszajn and all κ^+-Aronszajn trees are special?

As pointed out by Rinot, by his result together with $\neg\square(\omega_2) + \neg\square_{\omega_2} + 2^{\aleph_1} = \aleph_2 \Longrightarrow \mathsf{AD}^{L(\mathbb{R})}$ (Schimmerling–Steel), if Yes then $\mathsf{AD}^{L(\mathbb{R})}$.

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Hauskaa päivän jatkoa!