Games of Length ω^2

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Slogan

The region of the consistency strength hierarchy between the theories

$$\mathrm{ZFC} + \{\text{“there are } n \text{ Woodin cardinals”}: \ n \in \mathbb{N}\}$$

and

$$\mathrm{ZFC}+ \text{ “there are infinitely many Woodin cardinals”}$$

resembles the region of the consistency strength hierarchy between PA and ZFC.

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2. increasing the collection of (Borel) games of length ω^2 that can be proved determined,
3. asserting the existence of less-weak jump operators.

Bounded Games

Theorem (Post, Simpson, folklore)

The following are equivalent over Recursive Comprehension:

1. *Arithmetical Comprehension, i.e., $L_{\omega+1}$-comprehension,*
2. *For every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, $x^{(n)}$ exists,*
3. *For every n, every $\mathbf{\Sigma}^0_1$ game of length n is determined.*

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Theorem (Neeman, Woodin)

The following are equivalent over ZFC:

1. *Projective determinacy, i.e., $L_1(\mathbb{R})$-determinacy,*
2. *For every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, $M_n^\sharp(x)$ exists,*
3. *For every n, every $\mathbf{\Sigma}^1_1$ game of length $\omega \cdot n$ is determined.*

Clopen Games

Theorem (Steel)

The following are equivalent over Recursive Comprehension:

1. *Clopen determinacy for games of length* ω,
2. *Arithmetical Transfinite Recursion, i.e.,* L_α*-comprehension for all countable* α,
3. *For every* $x \in \mathbb{R}$ *and every countable* α, $x^{(\alpha)}$ *exists.*

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Theorem

The following are equivalent over ZFC:

1. *Clopen determinacy for games of length* ω^2,
2. σ*-projective determinacy, i.e.,* $L_{\omega_1}(\mathbb{R})$*-determinacy,*
3. *For every* $x \in \mathbb{R}$ *and every countable* α, $N^\sharp_\alpha(x)$ *exists.*

Clopen Games

We will come back to clopen games of length ω^2 later. A precursor to this theorem is:

Theorem (with S. Müller and P. Schlicht)

The following are equivalent over ZFC*:*

1. σ*-projective determinacy,*
2. *Determinacy for simple clopen games of length* ω^2*,*
3. *Determinacy for simple* σ*-projective games of length* ω^2*.*

Open Games

Theorem (Solovay)

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The following are equivalent over ZFC*:*

1. $\boldsymbol{\Sigma}^0_1$*-determinacy for games of length* ω^2*,*
2. *there is an admissible set containing* $\mathbb{R}$ *and satisfying* AD*.*

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Definition

Given a set A, let A^+ denote the intersection of all admissible sets containing A. A set is Π_1^+-reflecting if for every Π_1 formula ψ, if $A^+ \models \psi(A)$, then there is $B \in A$ such that $B^+ \models \psi(B)$.

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Theorem (Martin)

The following are equivalent over KP + *Separation:*

1. *Borel determinacy for games of length* ω,
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Theorem

The following are equivalent over ZFC:

1. *Borel determinacy for games of length* ω^2,
2. *for every countable* α, *there is a* β *such that* $L_\beta(\mathbb{R})$ *satisfies* "V_α *exists"* + AD,
3. *for every countable* α, *there is a countably iterable extender model satisfying Z +* "V_α *exists" + "there are infinitely many Woodin cardinals."*

Back to the beginning

Theorem (Neeman, Woodin)

The following are equivalent over ZFC*:*

1. *Projective determinacy, i.e.,* $L_1(\mathbb{R})$*-determinacy,*
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Theorem (with S. Müller)

The following are equiconsistent:

1. *Projective determinacy for games of length* ω^2*,*
2. $\mathrm{ZFC} + \{$ *"there are* $\omega + n$ *Woodin cardinals":* $n \in \mathbb{N}\}$*,*
3. $\mathrm{ZF} + \mathrm{AD} + \{$ *"there are* n *Woodin cardinals":* $n \in \mathbb{N}\}$*.*

The direction (2) to (1) is due to Neeman.

Clopen games

Now that the stage has been set, let us go back to the theorem on clopen games.

Theorem

Suppose that σ-projective games of length ω are determined. Then, all clopen games of length ω^2 are determined.

Recall that the σ-projective sets are the smallest σ-algebra containing the open sets and closed under continuous images and are the sets of reals in $L_{\omega_1}(\mathbb{R})$.

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Recall that the σ-projective sets are the smallest σ-algebra containing the open sets and closed under continuous images and are the sets of reals in $L_{\omega_1}(\mathbb{R})$.
Recall also that the converse follows from the joint theorem with S. Müller and P. Schlicht.

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- Let $A \subset \mathbb{R} \times \mathbb{R}$ be clopen and write A_x for the set of all y such that $(x, y) \in A$.

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- Let $A \subset \mathbb{R} \times \mathbb{R}$ be clopen and write A_x for the set of all y such that $(x, y) \in A$.
- Let $\Game^{\mathbb{R}} A = \{x :$ Player I has a winning strategy in the game on $\mathbb{R}$ with payoff $A_x\}$.

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  Player I has a winning strategy in the game on $\mathbb{R}$ with payoff $A_x\}$.
- Let $\Game^{\mathbb{R}} \mathbf{\Delta}^0_1 = \{\Game^{\mathbb{R}} A : \; A \text{ is clopen}\}$.

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Lemma

$\Game^{\mathbb{R}} \mathbf{\Delta}^0_1 \subset L_{\omega_1}(\mathbb{R})$.

Proof of the theorem from the lemma

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Suppose first that the lemma holds.

- Let A be a clopen set and consider the game of length ω^2 on $\mathbb{N}$ with payoff A. We adapt an argument of Blass.

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Suppose first that the lemma holds.

- Let A be a clopen set and consider the game of length ω^2 on $\mathbb{N}$ with payoff A. We adapt an argument of Blass.
- Consider the following game:

$$\begin{array}{c|ccccc} I & \sigma_0 & & \sigma_1 & & \dots \\ II & & \tau_0 & & \tau_1 & \dots \end{array} \tag{1}$$

  Here, players I and II take turns playing reals coding strategies for Gale-Stewart games. Player I wins if

$$(\sigma_0 * \tau_0, \sigma_1 * \tau_1, \dots) \in A,$$

  where $\sigma * \tau$ denotes the result of facing off the strategies σ and τ.

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- This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem.

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- Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff A.

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- This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem.
- Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff A.
- Suppose instead that Player II has a winning strategy; we claim she has one in the long game.

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- Suppose instead that Player II has a winning strategy.
- We will construct a strategy τ for Player II in the long game with the property that every partial play by τ is not a losing play for Player II. Since the game is clopen, there can be no full play in which the winner of the game has not been decided, so τ will be a winning strategy.

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- The strategy is constructed by blocks; first, we define it for plays of finite length.

Proof of the theorem from the lemma

- Given $x \in \mathbb{R}$, one may consider the following variant G_x of (1):

$$\begin{array}{c|ccccc} I & \sigma_1 & & \sigma_2 & & \ldots \\ II & & \tau_1 & & \tau_2 & \ldots \end{array}$$

Here, Player I wins if, and only if,

$$(x, \sigma_1 * \tau_1, \ldots) \in A;$$

otherwise, Player II wins.

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- This is also a clopen game, so the set

$$W = \{x \in \mathbb{R} : \text{ Player I has a winning strategy in } G_x\}$$

  belongs to $\Game^{\mathbb{R}} \mathbf{\Delta}^0_1$, and thus to $L_{\omega_1}(\mathbb{R})$, by the lemma. By hypothesis,

$$L_{\omega_1}(\mathbb{R}) \models \mathrm{AD},$$

  and so W is determined.

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- Player I cannot have a winning strategy, for otherwise it could have been used as a first move to obtain a winning strategy in (1).
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- Thus, Player II has a winning strategy in W.
- This will provide the restriction of τ to the first ω-many moves.
- Given the first ω-many moves, say, a, one repeats the argument above to obtain the restriction of τ to moves of length $\omega \cdot 2$ extending a. Eventually, one defines the response of τ to every $b \in \mathbb{N}^{<\omega^2}$, as desired.

Proof of the lemma

Lemma

$\Game^{\mathbb{R}} \mathbf{\Delta}^0_1 \subset L_{\omega_1}(\mathbb{R})$.

- Let $A \subset \mathbb{R} \times \mathbb{R}$ be clopen. For each $x \in \mathbb{R}$, there is a game of length ω with moves in $\mathbb{R}$ given by A_x. Let us identify this game with A_x. We shall show that $\Game^{\mathbb{R}} A \in L_{\omega_1}(\mathbb{R})$.

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- For every $x \in \mathbb{R}$, we define $T_x = \Big\{ t \in \mathbb{R}^{<\mathbb{N}} : \exists y \in \mathbb{R}^{\mathbb{N}} \, \exists z \in \mathbb{R}^{\mathbb{N}} \, \big(t \sqsubset y \wedge t \sqsubset z \wedge (x, y) \in A \wedge (x, z) \notin A \big) \Big\}$.
- Thus, T_x is the set of "contested" positions in A_x.

Proof of the lemma

Lemma

$\Game^{\mathbb{R}}\boldsymbol{\Delta}^0_1 \subset L_{\omega_1}(\mathbb{R})$.

- We define a binary relation on $\mathbb{R}^2$ by

  $(x, y) \prec (w, z)$ if, and only if, $y \in \mathbb{R}^{<\mathbb{N}} \wedge x = w \wedge z \in T_w \wedge z \sqsubset y$.

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- Since A is clopen, for every $x \in \mathbb{R}$ and every $y \in \mathbb{R}^{\mathbb{N}}$ there is some $n \in \mathbb{N}$ such that for every $z \in \mathbb{R}^{\mathbb{N}}$,

$$y \upharpoonright n = z \upharpoonright n \text{ implies } (y \in A_x \leftrightarrow z \in A_x).$$

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  It follows that $\prec$ is wellfounded, so it has a rank function, ρ. Since $\prec$ is analytic, ρ is bounded below ω_1, say, by η.

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- Let us write

$$y \prec_x z \text{ if, and only if, } (x, y) \prec (x, z)$$

and denote by ρ_x the associated rank function.

Proof of the lemma

Lemma

$\Game^{\mathbb{R}}\boldsymbol{\Delta}^0_1 \subset L_{\omega_1}(\mathbb{R})$.

- Define:

$$W_0(x) = \Big\{a \in \mathbb{R}^{<\mathbb{N}} : \exists y \in \mathbb{R}\, \forall z \in \mathbb{R} \Big(a^\frown y^\frown z \notin T_x \wedge \exists w \in \mathbb{R}^{\mathbb{N}} \left(a^\frown y^\frown z \sqsubset w \wedge (x, w) \in A\right)\Big)\Big\};$$

$$W_\alpha(x) = \Big\{a \in \mathbb{R}^{<\mathbb{N}} : \exists y \in \mathbb{R}\, \forall z \in \mathbb{R} \left(a^\frown y^\frown z \in \bigcup_{\xi<\alpha} W_\xi(x)\right)\Big\};$$

$$W_\infty(x) = \bigcup_{\alpha \in \mathrm{Ord}} W_\alpha(x).$$

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$$W_\infty(x) = \bigcup_{\alpha \in \mathrm{Ord}} W_\alpha(x).$$

For a partial play a of even length, Player I has a winning strategy from a in A_x if, and only if, $a \in W_\infty(x)$.

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Lemma

$\Game^{\mathbb{R}} \boldsymbol{\Delta}^0_1 \subset L_{\omega_1}(\mathbb{R})$.

- Let us refer to the least ξ such that $y \in W_\xi(x)$, if any, as the *weight* of y and denote it by $w_x(y)$.

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- Let us refer to the least ξ such that $y \in W_\xi(x)$, if any, as the *weight* of y and denote it by $w_x(y)$.
- If a has weight ξ, then any extension of a of smaller weight has smaller rank in $\prec_x$.

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- If a has weight ξ, then any extension of a of smaller weight has smaller rank in $\prec_x$.
- By induction on the weight, it follows that for every $a \in W_\infty(x)$, $w_x(a) \leq \rho_x(a)$.

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- If a has weight ξ, then any extension of a of smaller weight has smaller rank in $\prec_x$.
- By induction on the weight, it follows that for every $a \in W_\infty(x)$, $w_x(a) \leq \rho_x(a)$.
- This implies

$$W_\infty(x) = W_\eta(x).$$

- Since the construction of $W_\eta(x)$ can be carried out within $L_{\omega_1}(\mathbb{R})$ uniformly in x, $\Game^{\mathbb{R}} A \in L_{\omega_1}(\mathbb{R})$, as desired.

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1. M is of class S_0 above δ if it has an initial segment which is active above δ;
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1. M is of class S_0 above δ if it has an initial segment which is active above δ;
2. M is of class $S_{\alpha+1}$ above δ if it has an initial segment N of class S_α above some $\delta_0 > \delta$ which is Woodin in N;
3. M is of class S_λ above δ if $\lambda < \omega_1^M$ and it has an active initial segment in all classes S_α above δ, for all $\alpha < \lambda$;

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Definition

Let $x \in \mathbb{R}$ and $\alpha < \omega_1^x$. Then, $N_\alpha^\sharp(x)$ is the unique least ω_1-iterable sound x-premouse of class S_α, if it exists.

The end

Thank you.