Games of Length $\omega^2$

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The region of the consistency strength hierarchy between the theories

\[ \text{ZFC} + \{ \text{“there are } n \text{ Woodin cardinals”: } n \in \mathbb{N} \} \]

and

\[ \text{ZFC}+ \text{ “there are infinitely many Woodin cardinals”} \]

resembles the region of the consistency strength hierarchy between \( \text{PA} \) and \( \text{ZFC} \).
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1. increasing the segments of $L(\mathbb{R})$ that can be proved to be determined,
2. increasing the collection of (Borel) games of length $\omega^2$ that can be proved determined,
3. asserting the existence of less-weak jump operators.
Theorem (Post, Simpson, folklore)

The following are equivalent over Recursive Comprehension:

1. **Arithmetical Comprehension**, i.e., $L_{\omega+1}$-comprehension,
2. For every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, $x^{(n)}$ exists,
3. For every $n$, every $\Sigma^0_1$ game of length $n$ is determined.
Bounded Games

Theorem (Post, Simpson, folklore)

The following are equivalent over Recursive Comprehension:

1. Arithmetical Comprehension, i.e., $L_{ω+1}$-comprehension,
2. For every $x ∈ ℝ$ and every $n ∈ ℕ$, $x^{(n)}$ exists,
3. For every $n$, every $Σ^0_1$ game of length $n$ is determined.

Theorem (Neeman, Woodin)

The following are equivalent over ZFC:

1. Projective determinacy, i.e., $L_1(ℝ)$-determinacy,
2. For every $x ∈ ℝ$ and every $n ∈ ℕ$, $M_n^♯(x)$ exists,
3. For every $n$, every $Σ^1_1$ game of length $ω · n$ is determined.
Theorem (Steel)

The following are equivalent over Recursive Comprehension:

1. Clopen determinacy for games of length \( \omega \),
2. Arithmetical Transfinite Recursion, i.e., \( L_\alpha \)-comprehension for all countable \( \alpha \),
3. For every \( x \in \mathbb{R} \) and every countable \( \alpha \), \( x^{(\alpha)} \) exists.
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1. Clopen determinacy for games of length $\omega$,
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3. For every $x \in \mathbb{R}$ and every countable $\alpha$, $x^{(\alpha)}$ exists.

Theorem
The following are equivalent over ZFC:

1. Clopen determinacy for games of length $\omega^2$,
2. $\sigma$-projective determinacy, i.e., $L_{\omega_1}(\mathbb{R})$-determinacy,
3. For every $x \in \mathbb{R}$ and every countable $\alpha$, $N_\alpha^\sharp(x)$ exists.
We will come back to clopen games of length $\omega^2$ later. A precursor to this theorem is:

**Theorem (with S. Müller and P. Schlicht)**

The following are equivalent over ZFC:

1. $\sigma$-projective determinacy,
2. Determinacy for simple clopen games of length $\omega^2$,
3. Determinacy for simple $\sigma$-projective games of length $\omega^2$. 

Theorem (Solovay)

The following are equivalent over KP:

1. $\Sigma^0_1$-determinacy for games of length $\omega$,
2. there is an admissible set containing $\mathbb{N}$. 

The following are equivalent over ZFC:

1. $\Sigma^0_1$-determinacy for games of length $\omega$,
2. there is an admissible set containing $\mathbb{R}$ and satisfying $\text{AD}$. 

Open Games

Theorem (Solovay)
The following are equivalent over $\text{KP}$:
1. $\Sigma_1^0$-determinacy for games of length $\omega$,
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Theorem
The following are equivalent over $\text{ZFC}$:
1. $\Sigma_1^0$-determinacy for games of length $\omega^2$,
2. there is an admissible set containing $\mathbb{R}$ and satisfying $\text{AD}$.
**Theorem (Solovay)***

*The following are equivalent over KP:*

1. \( \Sigma^0_2 \)-determinacy for games of length \( \omega \),
2. there is a \( \Sigma^1_1 \)-reflecting ordinal.

---

**Definition**

Given a set \( A \), let \( A^+ \) denote the intersection of all admissible sets containing \( A \). A set is \( \Pi^+_1 \)-reflecting if for every \( \Pi^1_1 \) formula \( \psi \), if \( A^+ \models \psi(A) \), then there is \( B \in A \) such that \( B^+ \models \psi(B) \).
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Theorem

The following are equivalent over ZFC:

1. $\Sigma^0_2$-determinacy for games of length $\omega^2$,
2. there is an admissible $\Pi^+_1$-reflecting set containing $\mathbb{R}$ and satisfying AD.
Theorem (Martin)

The following are equivalent over KP + Separation:

1. Borel determinacy for games of length ω,
2. for every $x \in \mathbb{R}$ and every countable $\alpha$, there is a $\beta$ such that $L_\beta[x]$ satisfies $Z + \text{"} V_\alpha \text{ exists."}$. 
Borel Games

Theorem (Martin)
The following are equivalent over KP + Separation:
1. Borel determinacy for games of length \( \omega \),
2. for every \( x \in \mathbb{R} \) and every countable \( \alpha \), there is a \( \beta \) such that \( L_\beta[x] \) satisfies \( Z + \text{“}V_\alpha \text{ exists}.”\)

Theorem
The following are equivalent over ZFC:
1. Borel determinacy for games of length \( \omega^2 \),
2. for every countable \( \alpha \), there is a \( \beta \) such that \( L_\beta(\mathbb{R}) \) satisfies \( \text{“}V_\alpha \text{ exists}” + \text{AD} \),
3. for every countable \( \alpha \), there is a countably iterable extender model satisfying \( Z + \text{“}V_\alpha \text{ exists}” + \text{“there are infinitely many Woodin cardinals.”} \)
Theorem (Neeman, Woodin)

The following are equivalent over ZFC:

1. Projective determinacy, i.e., $L_1(\mathbb{R})$-determinacy,
2. For every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, $M_n^\#$ exists,
3. For every $n$, every $\Sigma^1_1$ game of length $\omega \cdot n$ is determined.

Theorem (with S. Müller)

The following are equiconsistent:

1. Projective determinacy for games of length $\omega^2$,
2. $\text{ZFC} + \{\text{there are } \omega+n \text{ Woodin cardinals}: n \in \mathbb{N}\}$,
3. $\text{ZF} + \text{AD} + \{\text{there are } n \text{ Woodin cardinals}: n \in \mathbb{N}\}$.

The direction (2) to (1) is due to Neeman.
Theorem (Neeman, Woodin)

The following are equivalent over ZFC:

1. Projective determinacy, i.e., $L_1(\mathbb{R})$-determinacy,
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Theorem (with S. Müller)

The following are equiconsistent:

1. Projective determinacy for games of length $\omega^2$,
2. ZFC $+$ \{“there are $\omega + n$ Woodin cardinals”$: n \in \mathbb{N}$$\}$,
3. ZF $+$ AD $+$ \{“there are $n$ Woodin cardinals”$: n \in \mathbb{N}$$\}$.

The direction (2) to (1) is due to Neeman.
Now that the stage has been set, let us go back to the theorem on clopen games.

**Theorem**

Suppose that $\sigma$-projective games of length $\omega$ are determined. Then, all clopen games of length $\omega^2$ are determined.

Recall that the $\sigma$-projective sets are the smallest $\sigma$-algebra containing the open sets and closed under continuous images and are the sets of reals in $L_{\omega_1}(\mathbb{R})$. Recall also that the converse follows from the joint theorem with S. Müller and P. Schlicht.
Clopen games

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Recall that the $\sigma$-projective sets are the smallest $\sigma$-algebra containing the open sets and closed under continuous images and are the sets of reals in $L_{\omega_1}(\mathbb{R})$.

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Let us sketch the proof of the theorem.
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- Let $A \subset \mathbb{R} \times \mathbb{R}$ be clopen and write $A_x$ for the set of all $y$ such that $(x, y) \in A$. 

### Lemma

$L_0^\omega (\mathbb{R}) \subset \mathcal{F}_{\omega}$. 

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Games of Length $\omega^2$
Let us sketch the proof of the theorem.

- Let $A \subset \mathbb{R} \times \mathbb{R}$ be clopen and write $A_x$ for the set of all $y$ such that $(x, y) \in A$.
- Let $\mathcal{R}A = \{x : $ Player I has a winning strategy in the game on $\mathbb{R}$ with payoff $A_x\}$. 
Let us sketch the proof of the theorem.

- Let $A \subset \mathbb{R} \times \mathbb{R}$ be clopen and write $A_x$ for the set of all $y$ such that $(x, y) \in A$.
- Let $\mathcal{D}^R A = \{x : \text{Player I has a winning strategy in the game on } \mathbb{R} \text{ with payoff } A_x\}$.
- Let $\mathcal{D}^R \Delta^0_1 = \{\mathcal{D}^R A : A \text{ is clopen}\}$.
Let us sketch the proof of the theorem.

- Let $A \subseteq \mathbb{R} \times \mathbb{R}$ be clopen and write $A_x$ for the set of all $y$ such that $(x, y) \in A$.
- Let $\partial^R A = \{x : \text{Player I has a winning strategy in the game on } \mathbb{R} \text{ with payoff } A_x\}$.
- Let $\partial^R \Delta^0_1 = \{\partial^R A : A \text{ is clopen}\}$.

Lemma

$\partial^R \Delta^0_1 \subset L_{\omega_1}(\mathbb{R})$. 
Lemma

$$\mathcal{D}^\mathbb{R} \Delta^0_1 \subset L_{\omega_1}(\mathbb{R}).$$

Suppose first that the lemma holds.

- Let $A$ be a clopen set and consider the game of length $\omega^2$ on $\mathbb{N}$ with payoff $A$. We adapt an argument of Blass.
Proof of the theorem from the lemma

Lemma

$$\mathcal{D}^R_{\Delta^0_1} \subset L_{\omega_1}(\mathbb{R}).$$

Suppose first that the lemma holds.

- Let $A$ be a clopen set and consider the game of length $\omega^2$ on $\mathbb{N}$ with payoff $A$. We adapt an argument of Blass.
- Consider the following game:

$$
\begin{array}{c|ccc}
& \sigma_0 & \sigma_1 & \ldots \\
I & & & \\
\hline
II & \tau_0 & \tau_1 & \ldots \\
\end{array}
$$

Here, players I and II take turns playing reals coding strategies for Gale-Stewart games. Player I wins if

$$\left(\sigma_0 \ast \tau_0, \sigma_1 \ast \tau_1, \ldots\right) \in A,$$

where $\sigma \ast \tau$ denotes the result of facing off the strategies $\sigma$ and $\tau$. 
Proof of the theorem from the lemma

Lemma

\[ \mathcal{E}^R \Delta^0_1 \subset L_{\omega_1}(\mathbb{R}). \]

Suppose first that the lemma holds.

- This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem.
Proof of the theorem from the lemma

Lemma

\[ \forall R \Delta^0_1 \subset L_{\omega_1}(R). \]

Suppose first that the lemma holds.

- This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem.
- Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff \( A \).
Proof of the theorem from the lemma

Lemma

$\mathcal{D}_{\mathbb{R}} \Delta^0_1 \subset L_{\omega_1}(\mathbb{R})$.

Suppose first that the lemma holds.

- This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem.
- Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff $A$.
- Suppose instead that Player II has a winning strategy; we claim she has one in the long game.
Lemma

\( \mathcal{D}^R_1 \Delta^0_1 \subset L_{\omega_1}(\mathbb{R}). \)

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Suppose first that the lemma holds.

- This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem.
- Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff $A$.

Suppose instead that Player II has a winning strategy. We will construct a strategy $\tau$ for Player II in the long game with the property that every partial play by $\tau$ is not a losing play for Player II. Since the game is clopen, there can be no full play in which the winner of the game has not been decided, so $\tau$ will be a winning strategy.

The strategy is constructed by blocks; first, we define it for plays of finite length.
Proof of the theorem from the lemma

Lemma

$$\mathcal{D}^R \Delta^0_1 \subset L_{\omega_1}(\mathbb{R}).$$

Suppose first that the lemma holds.

- This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem.
- Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff $A$.
- Suppose instead that Player II has a winning strategy.
Proof of the theorem from the lemma

Lemma

\( \mathcal{D}^{\mathbb{R}} \Delta^0_1 \subset L_{\omega_1}(\mathbb{R}). \)

Suppose first that the lemma holds.

- This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem.
- Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff \( A \).
- Suppose instead that Player II has a winning strategy.
- We will construct a strategy \( \tau \) for Player II in the long game with the property that every partial play by \( \tau \) is not a losing play for Player II. Since the game is clopen, there can be no full play in which the winner of the game has not been decided, so \( \tau \) will be a winning strategy.
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Suppose first that the lemma holds.

- This is a clopen game on reals, so it is determined by the Gale-Stewart Theorem.
- Clearly, if Player I has a winning strategy in this game, then she has one in the long game with payoff $A$.
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- We will construct a strategy $\tau$ for Player II in the long game with the property that every partial play by $\tau$ is not a losing play for Player II. Since the game is clopen, there can be no full play in which the winner of the game has not been decided, so $\tau$ will be a winning strategy.
- The strategy is constructed by blocks; first, we define it for plays of finite length.
Proof of the theorem from the lemma

Given $x \in \mathbb{R}$, one may consider the following variant $G_x$ of (1):

$$
\begin{array}{c|cccc}
 & \sigma_1 & \sigma_2 & \cdots \\
\hline
I & \\
\hline
\hline
II & \tau_1 & \tau_2 & \cdots \\
\end{array}
$$

Here, Player I wins if, and only if,

$$(x, \sigma_1 \ast \tau_1, \ldots) \in A;$$

otherwise, Player II wins.
Proof of the theorem from the lemma

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<table>
<thead>
<tr>
<th></th>
<th>( \sigma_1 )</th>
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<tbody>
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<td></td>
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</tbody>
</table>

Here, Player I wins if, and only if,

\[
(x, \sigma_1 * \tau_1, \ldots) \in A;
\]

otherwise, Player II wins.

This is also a clopen game, so the set

\[
W = \{ x \in \mathbb{R} : \text{Player I has a winning strategy in } G_x \}
\]

belongs to \( \mathcal{O}^\mathbb{R} \Delta^0_1 \), and thus to \( L_{\omega_1}(\mathbb{R}) \), by the lemma. By hypothesis,

\[
L_{\omega_1}(\mathbb{R}) \models \text{AD},
\]

and so \( W \) is determined.
Proof of the theorem from the lemma

- Player I cannot have a winning strategy, for otherwise it could have been used as a first move to obtain a winning strategy in (1).
- Thus, Player II has a winning strategy in $\mathcal{W}$. 
Proof of the theorem from the lemma

- Player I cannot have a winning strategy, for otherwise it could have been used as a first move to obtain a winning strategy in (1).
- Thus, Player II has a winning strategy in \( W \).
- This will provide the restriction of \( \tau \) to the first \( \omega \)-many moves.
Proof of the theorem from the lemma

- Player I cannot have a winning strategy, for otherwise it could have been used as a first move to obtain a winning strategy in (1).
- Thus, Player II has a winning strategy in $W$.
- This will provide the restriction of $\tau$ to the first $\omega$-many moves.
- Given the first $\omega$-many moves, say, $a$, one repeats the argument above to obtain the restriction of $\tau$ to moves of length $\omega \cdot 2$ extending $a$. Eventually, one defines the response of $\tau$ to every $b \in \mathbb{N}^{<\omega^2}$, as desired.
Proof of the lemma

**Lemma**

\( \mathcal{D}^R \Delta_1^0 \subset L_{\omega_1}(\mathbb{R}) \).

- Let \( A \subset \mathbb{R} \times \mathbb{R} \) be clopen. For each \( x \in \mathbb{R} \), there is a game of length \( \omega \) with moves in \( \mathbb{R} \) given by \( A_x \). Let us identify this game with \( A_x \). We shall show that \( \mathcal{D}^R A \in L_{\omega_1}(\mathbb{R}) \).
Proof of the lemma

Lemma

$\Delta^0_1 \subset L_{\omega_1}(\mathbb{R})$.

Let $A \subset \mathbb{R} \times \mathbb{R}$ be clopen. For each $x \in \mathbb{R}$, there is a game of length $\omega$ with moves in $\mathbb{R}$ given by $A_x$. Let us identify this game with $A_x$. We shall show that $\exists^\mathbb{R} A \in L_{\omega_1}(\mathbb{R})$.

For every $x \in \mathbb{R}$, we define $T_x = \left\{ t \in \mathbb{R}^{<N} : \exists y \in \mathbb{R}^N \exists z \in \mathbb{R}^N (t \sqsubset y \land t \sqsubset z \land (x, y) \in A \land (x, z) \notin A) \right\}$.

Thus, $T_x$ is the set of "contested" positions in $A_x$. 

Proof of the lemma

Lemma

$\forall \mathbb{R} \Delta^0_1 \subset L_{\omega_1}(\mathbb{R})$.

- We define a binary relation on $\mathbb{R}^2$ by

$$(x, y) \prec (w, z) \text{ if, and only if, } y \in \mathbb{R}^{<\mathbb{N}} \wedge x = w \wedge z \in T_w \wedge z \sqsubseteq y.$$
Proof of the lemma

Lemma

\[ \mathcal{D}^\mathbb{R} \Delta^0_1 \subseteq L_{\omega_1}(\mathbb{R}). \]

- We define a binary relation on \( \mathbb{R}^2 \) by
  \[(x, y) \prec (w, z) \text{ if, and only if, } y \in \mathbb{R}^{< \mathbb{N}} \land x = w \land z \in T_w \land z \sqsubseteq y.\]

- Since \( A \) is clopen, for every \( x \in \mathbb{R} \) and every \( y \in \mathbb{R}^\mathbb{N} \) there is some \( n \in \mathbb{N} \) such that for every \( z \in \mathbb{R}^\mathbb{N} \),
  \[y \upharpoonright n = z \upharpoonright n \text{ implies } (y \in A_x \leftrightarrow z \in A_x).\]
Proof of the lemma

Lemma

$\mathcal{D}_1^\mathcal{R} \Delta_1^0 \subset L_{\omega_1}(\mathbb{R})$.

- We define a binary relation on $\mathbb{R}^2$ by
  
  $$(x, y) \prec (w, z) \text{ if, and only if, } y \in \mathbb{R}^N \land x = w \land z \in T_w \land z \sqsubseteq y.$$ 

- Since $A$ is clopen, for every $x \in \mathbb{R}$ and every $y \in \mathbb{R}^N$ there is some $n \in \mathbb{N}$ such that for every $z \in \mathbb{R}^N$, 

  $$y \upharpoonright n = z \upharpoonright n \text{ implies } (y \in A_x \leftrightarrow z \in A_x).$$

  It follows that $\prec$ is wellfounded, so it has a rank function, $\rho$. Since $\prec$ is analytic, $\rho$ is bounded below $\omega_1$, say, by $\eta$. 

Proof of the lemma

**Lemma**
\( \mathcal{D}^\mathbb{R}_1 ^\mathbb{N} \subseteq L_{\omega_1}(\mathbb{R}) \).

- We define a binary relation on \( \mathbb{R}^2 \) by
  \[(x, y) \prec (w, z) \text{ if, and only if, } y \in \mathbb{R}^\mathbb{N} \land x = w \land z \in T_w \land z \sqsubseteq y.\]

- Since \( A \) is clopen, for every \( x \in \mathbb{R} \) and every \( y \in \mathbb{R}^\mathbb{N} \) there is some \( n \in \mathbb{N} \) such that for every \( z \in \mathbb{R}^\mathbb{N} \),
  \[y \upharpoonright n = z \upharpoonright n \implies (y \in A_x \iff z \in A_x).\]
  It follows that \( \prec \) is wellfounded, so it has a rank function, \( \rho \). Since \( \prec \) is analytic, \( \rho \) is bounded below \( \omega_1 \), say, by \( \eta \).

- Let us write
  \[y \prec_x z \text{ if, and only if, } (x, y) \prec (x, z)\]
  and denote by \( \rho_x \) the associated rank function.
Proof of the lemma

**Lemma**

\( \forall \mathcal{R} \, \Delta_1^0 \subset L_{\omega_1}(\mathcal{R}). \)

Define:

\[
W_0(x) = \left\{ a \in \mathcal{R}^{<\mathbb{N}} : \exists y \in \mathcal{R} \, \forall z \in \mathcal{R} \left( a \upharpoonright y \upharpoonright z \not\in T_x \land \exists w \in \mathcal{R}^{\mathbb{N}} (a \upharpoonright y \upharpoonright z \sqsubset w \land (x, w) \in A) \right) \right\};
\]

\[
W_\alpha(x) = \left\{ a \in \mathcal{R}^{<\mathbb{N}} : \exists y \in \mathcal{R} \, \forall z \in \mathcal{R} \left( a \upharpoonright y \upharpoonright z \in \bigcup_{\xi < \alpha} W_\xi(x) \right) \right\};
\]

\[
W_\infty(x) = \bigcup_{\alpha \in \text{Ord}} W_\alpha(x).
\]
Proof of the lemma

**Lemma**

\( \forall \mathbb{R} \Delta_1^0 \subset L_{\omega_1}(\mathbb{R}). \)

- Define:

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  W_0(x) = \left\{ a \in \mathbb{R}^{<\omega} : \exists y \in \mathbb{R} \forall z \in \mathbb{R} \left( a \upharpoonright y \upharpoonright z \not\in T_x \land \right. \right. \\
  \left. \left. \exists w \in \mathbb{R}^{\omega} \left( a \upharpoonright y \upharpoonright z \sqsubseteq w \land (x, w) \in A \right) \right) \right\};
  \]

  \[
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  \]

  \[
  W_{\infty}(x) = \bigcup_{\alpha \in \text{Ord}} W_\alpha(x).
  \]

For a partial play \( a \) of even length, Player I has a winning strategy from \( a \) in \( A_x \) if, and only if, \( a \in W_{\infty}(x) \).
Proof of the lemma

Lemma

$\mathcal{D}^R \Delta_1^0 \subset L_{\omega_1}(\mathbb{R})$.

- Let us refer to the least $\xi$ such that $y \in W_\xi(x)$, if any, as the weight of $y$ and denote it by $w_x(y)$. 
Proof of the lemma

Lemma

$$\mathcal{D}^R_{\Delta^0_1} \subset L_{\omega_1}(\mathbb{R}).$$

- Let us refer to the least $\xi$ such that $y \in W_\xi(x)$, if any, as the *weight* of $y$ and denote it by $w_x(y)$.
- If $a$ has weight $\xi$, then any extension of $a$ of smaller weight has smaller rank in $\prec_x$.

By induction on the weight, it follows that for every $a \in W_\omega(x)$, $w_x(a) \leq \rho_x(a)$. This implies $W_\omega(x) = W_\eta(x)$. Since the construction of $W_\eta(x)$ can be carried out within $L_{\omega_1}(\mathbb{R})$ uniformly in $x$, we have $\mathcal{D}^R_{\Delta^0_1} \subset L_{\omega_1}(\mathbb{R})$, as desired.
Proof of the lemma

Lemma

$\mathcal{D}^R \Delta_1^0 \subset L_{\omega_1}(\mathbb{R})$.

- Let us refer to the least $\xi$ such that $y \in W_\xi(x)$, if any, as the weight of $y$ and denote it by $w_x(y)$.
- If $a$ has weight $\xi$, then any extension of $a$ of smaller weight has smaller rank in $\prec_x$.
- By induction on the weight, it follows that for every $a \in W_\infty(x)$, $w_x(a) \leq \rho_x(a)$. 
Proof of the lemma

Lemma

$\mathcal{D}^R \Delta^0_1 \subseteq L_{\omega_1}(\mathbb{R})$.

- Let us refer to the least $\xi$ such that $y \in W_\xi(x)$, if any, as the weight of $y$ and denote it by $w_x(y)$.
- If $a$ has weight $\xi$, then any extension of $a$ of smaller weight has smaller rank in $\prec_x$.
- By induction on the weight, it follows that for every $a \in W_\infty(x)$, $w_x(a) \leq \rho_x(a)$.
- This implies
  $$W_\infty(x) = W_\eta(x).$$
- Since the construction of $W_\eta(x)$ can be carried out within $L_{\omega_1}(\mathbb{R})$ uniformly in $x$, $\mathcal{D}^R A \in L_{\omega_1}(\mathbb{R})$, as desired.
We also mentioned:

**Theorem**

The following are equivalent:

1. $\sigma$-projective determinacy,
2. for every $\alpha$, $N^\#_{\alpha}(x)$ exists for almost every $x$. 

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**Theorem**

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Models of class $S_\alpha$

Definition

Let $M$ be a countable, $\omega_1$-iterable extender model of some fragment of ZFC.

1. $M$ is of class $S_0$ above $\delta$ if it has an initial segment which is active above $\delta$;

2. $M$ is of class $S_\alpha$ above $\delta$ if it has an initial segment of class $S_\alpha$ above some $\delta_0 > \delta$ which is Woodin in $N$;

3. $M$ is of class $S_\lambda$ above $\delta$ if $\lambda < \omega_1^M$ and it has an active initial segment in all classes $S_\alpha$ above $\delta$, for all $\alpha < \lambda$;

4. $M$ is of class $S_\alpha$ if it is of class $S_\alpha$ above $0$. 

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**Definition**

Let $x \in \mathbb{R}$ and $\alpha < \omega_1^x$. Then, $N^x_\alpha(x)$ is the unique least $\omega_1$-iterable sound $x$-premouse of class $S_\alpha$, if it exists.
Thank you.