Definability of maximal discrete sets

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Arctic Set Theory 3, Kilpisjärvi
January 28, 2017

Outline

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Discrete sets

Let R be a binary symmetric relation on a set X.

Definition

We say a set $A \subseteq X$ is **discrete** (w.r.t. R) $\iff$ no two distinct elements x, y of A are R-related.

Definition

We call such a set **maximal discrete** (w.r.t. R; short an R-**mds**) if it is not a proper subset of any discrete set.

Let $\mathrm{span}_R(A) = A \cup \{x \in X \mid (\exists a \in A)\, a\, R\, x\}$.

Then A is maximal discrete *iff* it is discrete and $\mathrm{span}_R(A) = X$.

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Discrete sets (non-binary)

Let X be a set and $R \subseteq [X]^{<\omega}$.

Definition

We say a set $A \subseteq X$ is **discrete** (w.r.t. R) $\iff$ $(\forall n > 1)\ [A]^n \cap R = \emptyset$.

The notion of R-**mds** is defined as before.

Maximal discrete sets exist by AC.

Our main interest is when R is an (effectively) Borel relation on an (effective) Polish space X. One can then ask whether it is possible that an R-**mds** is *definable* or more precisely, where such sets first appear in the (lightface) projective hierarchy.

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Short excursion: Irregular sets of reals

We think of maximal discrete sets as a type of *irregular set* of reals.

Some classical regularity properties:

- Lebesgue measurability
- Baire property
- being completely Ramsey (Baire property with respect to the Ellentuck-topology, in $[\omega]^\omega$)

How complicated must a set of reals be in order to be *irregular*?

- analytic sets can usually be shown to be regular
- In $\mathbf{L}$, there are Δ^1_2 irregular sets
- Under large cardinals, all projective sets are regular
- between these extremes, one can obtain lots of independence results via forcing (some requiring smaller large cardinals)

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Instances of maximal discrete sets

Binary

- Transversals for equivalence relations
- Mad families
- Maximal eventually different families
- Maximal independent families of sets (or of functions)
- Maximal orthogonal families of measures (mofs)

Higher arity

- Hamel basis (basis of $\mathbb{R}$ over $\mathbb{Q}$)
- Maximal cofinitary groups (mcgs)

This talk is about mofs, mcgs and Hamel bases.

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Interaction between different notions

Existence of one type of irregular or maximal discrete set can entail the existence of another.

- If there is a projective Hamel basis, there is a projective Vitali set.
- "Every Σ^1_2 set is Lebesgue measurable" $\Rightarrow$ "every Σ^1_2 set has the property of Baire" (Bartoszynsky 1984).

More often, one can show no such interaction occurs:

Theorem (Shelah 1984)

"Every projective set has the property of Baire" $\not\Rightarrow$ "Every projective set is Lebesgue measurable"

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Outline

Cofinitary groups

- Work in the space $X = S_\infty$, the group of bijections from $\mathbb{N}$ to itself (permutations).
- $\mathrm{id}_\mathbb{N}$ is the identity function on $\mathbb{N}$, the neutral element of S_∞.

Definition

We say $g \in S_\infty$ is *cofinitary* $\iff$

$$\{n \in \mathbb{N} \mid g(n) = n\} \text{ is finite.}$$

$\mathcal{G} \leq S_\infty$ is *cofinitary* $\iff$ every $g \in \mathcal{G} \setminus \{\mathrm{id}_\mathbb{N}\}$ is cofinitary.

A maximal cofinitary group is maximal R-discrete set, where

$$\{g_0, \ldots, g_n\} \in R \iff \langle g_0, \ldots, g_n \rangle^{S_\infty} \text{ is not cofinitary.}$$

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Definability of mcgs

Theorem (Kastermans)

No mcg can be K_σ.

Some history:

- Gao-Zhang: If $\mathbf{V} = \mathbf{L}$, there is a mcg with a Π^1_1 set of generators.
- Kastermans: If $\mathbf{V} = \mathbf{L}$, there is a Π^1_1 mcg.

Theorem (Fischer-S.-Törnquist, 2015)

If $\mathbf{V} = \mathbf{L}$, there is a Π^1_1 mcg which remains maximal after adding any number of Cohen reals.

Surprisingly, and in contrast to classical irregular sets:

Theorem (Horowitz-Shelah, 2016)

(ZF) There is a Borel maximal cofinitary group.

By Σ^1_2 absoluteness, a Borel mcg remains maximal in any outer model.

They also claim they will show there is a closed mcg in a future paper.

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Adding a generic cofinitary group

Theorem (Zhang)

Let $\mathcal{G}$ be a cofinitary group. There is a forcing $\mathbb{P}_{\mathcal{G}}$ which adds a generic permutation σ such that

1. *$\mathcal{G}' = \langle \mathcal{G}, \sigma \rangle$ is cofinitary,*
2. *$\mathcal{G}'$ is maximal with respect to the ground model: For any $\tau \in V \setminus \mathcal{G}$, $\langle \mathcal{G}', \tau \rangle$ is not cofinitary.*

We adapted this forcing so that given an arbitrary $z \in 2^{\omega}$ in addition, every new group element *codes* z:

Theorem (Fischer-Törnquist-S. 2015)

Let $\mathcal{G}$ be a cofinitary group and $z \in 2^{\omega}$. There is a forcing $\mathbb{P}_{\mathcal{G},z}$ which adds a generic permutation σ such that in addition to 1 and 2 above

3. *z is computable from any $x \in \mathcal{G}' \setminus \mathcal{G}$.*

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Let $\mathcal{G}$ be a cofinitary group and $z \in 2^\omega$. There is a forcing $\mathbb{P}_{\mathcal{G},z}$ which adds a generic permutation σ such that in addition to 1 and 2 above

3. *z is computable from any $x \in \mathcal{G}' \setminus \mathcal{G}$.*

Adding a generic cofinitary group

Theorem (Zhang)

Let $\mathcal{G}$ be a cofinitary group. There is a forcing $\mathbb{P}_{\mathcal{G}}$ which adds a generic permutation σ such that

1. $\mathcal{G}' = \langle \mathcal{G}, \sigma \rangle$ *is cofinitary,*
2. $\mathcal{G}'$ *is maximal with respect to the ground model: For any* $\tau \in V \setminus \mathcal{G}$, $\langle \mathcal{G}', \tau \rangle$ *is not cofinitary.*

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A Cohen-indestructible Π^1_1 maximal cofinitary group

The group is constructed by recursion, reproving Kasterman's Theorem and imitating Miller's classical construction of Π^1_1 **mds.**

1. Assume we have $\{\sigma_\nu \mid \nu < \xi\} = \mathcal{G}_\xi$ where $\xi < \omega_1$.
2. Let $\eta < \omega_1$ be least such that $\mathcal{G}_\xi \in \mathbf{L}_\eta$.
3. We may demand that moreover there is a surjection from ω onto $\mathbf{L}_\eta$ which is definable in $\mathbf{L}_\eta$.
4. Use this to code $\mathbf{L}_\eta$ canonically into a real z.
5. Let σ_ξ be the $\leq_{\mathbf{L}}$-least generic over $\mathbf{L}_\eta$ for $\mathbb{P}_{\mathcal{G}_\xi, z}$.

The "natural" formula expressing membership in $\mathcal{G} = \bigcup_{\xi<\omega_1} \mathcal{G}_\xi$ is Σ_1 resp. Σ^1_2. It can be replaced by a Π^1_1 formula because each $\sigma \in \mathcal{G}$ knows via z a witness to the leading existential quantifier.

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Outline

Orthogonality of measures

- Let $P(2^\omega)$ be the set of Borel probability measures on 2^ω.
- Note that $P(2^\omega)$ is an effective Polish space.
- Two measures $\mu, \nu \in P(2^\omega)$ are said to be orthogonal, written

  $$\mu \perp \nu$$

  exactly if there is a Borel set $A \subseteq 2^\omega$ such that

  $$\mu(A) = 1$$

  and

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  This is an arithmetical relation.
- A maximal discrete set w.r.t. $\not\perp$ is called a *maximal orthogonal family of measures* (or short, **mof**).

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History of maximal orthogonal families

Question (Mauldin, circa 1980)

Can a **mof** in $P(2^\omega)$ be analytic?

The answer turned out to be 'no':

Theorem (Preiss-Rataj, 1985)

There is no analytic ***mof*** *in* $P(2^\omega)$.

This is optimal, in a sense:

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In $\mathbf{L}$, *there is a* Π^1_1 ***mof*** *in* $P(2^\omega)$.

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If there is a Σ^1_2 ***mof*** *in* $P(2^\omega)$, *there is a* Π^1_1 ***mof***.

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Can definable mofs survive forcing?

Mofs are fragile creatures:

1. Adding any real destroys maximality of **mofs** from the groundmodel (observed by Ben Miller; not restricted to forcing extensions)
2. Using methods reminiscent of Hjorth's theory of turbulency, one can show there is no Σ^1_2 **mof** whenever there exists a real of the following type over **L**:
   - A Cohen real (Fischer-Törnquist, 2009)
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A general theorem for Σ^1_1 relations

Theorem (S. 2016)

*Let R be a binary symmetric Σ^1_1 relation on an effective Polish space X. If $\bar{s}$ is generic for iterated Sacks forcing over $\mathbf{L}$, there is a Δ^1_2 R-**mds** in $\mathbf{L}[\bar{s}]$.*

Note we are always referring to the *lightface* (effective) hierarchy.

As existence of a Σ^1_2 **mof** implies existence of a Π^1_1 **mof**, we obtain a strong negative answer to the previous question:

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Proof sketch (for a single Sacks real and Borel R).

Assume R is a symmetric Δ^1_1 relation on ω^ω and $\mathbf{L}[s]$ is a Sacks extension of $\mathbf{L}$.

Recall that Sacks forcing $\mathbb{S}$ is the set of *perfect trees* $p \subseteq 2^{<\omega}$, ordered by inclusion and $[p]$ is the set of branches through p.

We need the following standard fact:

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Any element of $\mathbf{L}[s] \cap \omega^\omega$ is equal to $f(s)$ for some continuous function $f\colon 2^\omega \to \omega^\omega$ with code in $\mathbf{L}$.

We also need the following theorem of Galvin:

Theorem (Galvin's Theorem)

Let $c\colon (2^\omega)^2 \to \{0,1\}$ be symmetric and Baire measurable. Then there is a perfect set $P \subseteq 2^\omega$ such that c is constant on $P^2 \setminus \mathrm{diag}$.

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Let $c\colon (2^\omega)^2 \to \{0,1\}$ be symmetric and Baire measurable. Then there is a perfect set $P \subseteq 2^\omega$ such that c is constant on $P^2 \setminus \mathrm{diag}$.

Proof sketch (for a single Sacks real and Borel R).

Assume R is a symmetric Δ^1_1 relation on ω^ω and $\mathbf{L}[s]$ is a Sacks extension of $\mathbf{L}$.

Recall that Sacks forcing $\mathbb{S}$ is the set of *perfect trees* $p \subseteq 2^{<\omega}$, ordered by inclusion and $[p]$ is the set of branches through p.

We need the following standard fact:

Fact

Any element of $\mathbf{L}[s] \cap \omega^\omega$ is equal to $f(s)$ for some continuous function $f\colon 2^\omega \to \omega^\omega$ with code in $\mathbf{L}$.

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A lemma: Complete and discrete conditions

Lemma

Suppose R is a Σ^1_1 symmetric binary relation on ω^ω, $p \in \mathbb{S}$, and $f \in C(2^\omega, \omega^\omega)$. There is $q \leq p$ such that one of the following holds:

1. *$f''[q]$ is R-discrete*
2. *$f''[q]$ is R-complete, i.e. any two elements of $f''[q]$ are R-related.*

Proof.

Apply Galvin's Theorem for the coloring on $[p]^2$ given by

$$c(x,y) = \begin{cases} 1 & \text{if } f(x) \mathrel{R} f(y), \\ 0 & \text{if } f(x) \not\mathrel{R} f(y). \end{cases}$$

□

Note: 1 is a Π^1_1 statement about q;
2 is a Π^1_2 statement about q, so both are absolute.

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Ideas for the proof (continued).

We also use the following well-known property of Sacks forcing, which can be seen as a special case of the previous:

Corollary

Say Φ is a Σ^1_1 (or Π^1_1) formula, $p \in \mathbb{S}$ and

$$p \Vdash \neg\Phi(\dot{s}).$$

Then there is $q \leq p$ such that

$$[q] \cap \{x \mid \Phi(x)\} = \emptyset.$$

Also note that $[q] \cap \{x \mid \Phi(x)\} = \emptyset$ is Π^1_1, hence absolute, and thus will also hold in the Sacks extension.

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Work in **L**. Let $\langle (p_\xi, f_\xi) \mid \xi < \omega_1 \rangle$ enumerate $\mathbb{S} \times \mathrm{C}(2^\omega, \omega^\omega)$.

By recursion, choose for each $\xi < \omega_1$ a tree $T_\xi \subseteq p_\xi$ such that in $\mathbf{L}[s]$, $\mathcal{A} = \bigcup\{[T_\xi] \mid \xi < \omega_1\}$ will be a **mds.** At stage $\xi < \omega_1$, suppose

$$p_\xi \Vdash f_\xi(\dot{s}) \notin \mathrm{span}_R \Big(\bigcup_{\nu<\xi} [T_\nu] \Big), \tag{1}$$

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It remains to show that $\mathcal{A}$ is maximal:

Towards a contradiction, suppose there is $x \in \mathbf{L}[s] \cap \omega^\omega$ and $x \notin \mathrm{span}_R(\mathcal{A})$.

We can pick

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Find the stage ξ when we considered (p, f), i.e. $(p, f) = (p_\xi, f_\xi)$. We found $q \leq p$ which was either complete or discrete.

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A Ramsey theoretic result about iterated Sacks forcing

One of the main ingredients for the general result for iterated Sacks forcing is an analogue of Galvin's theorem.

- Let $\mathbb{P}$ be a countable support iteration of Sacks forcing.
- On a dense set of $\bar{p} \in \mathbb{P}$, we can define $[\bar{p}]$ as a perfect subspace of $(2^\omega)^{\mathrm{supp}(\bar{p})}$ in a meaningful way ($\mathrm{supp}(\bar{p})$ is the support of $\bar{p}$).

Question:

Is there for every $\bar{p} \in \mathbb{P}$ and every *nice* symmetric $c \colon [\bar{p}]^2 \to \{0, 1\}$ some $\bar{q} \in \mathbb{P}$, $\bar{q} \leq \bar{p}$ such that c is constant on $[\bar{q}]^2 \setminus \mathrm{diag}$?

'Yes', if c is *continuous* on $[\bar{p}]^2 \setminus \mathrm{diag}$ (Geschke-Kojman-Kubiś-Schipperus)

For more complicated c, there are combinatorial obstructions to such a straightforward generalization.

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Is there for every $\bar{p} \in \mathbb{P}$ and every *nice* symmetric $c\colon [\bar{p}]^2 \to \{0,1\}$ some $\bar{q} \in \mathbb{P}$, $\bar{q} \leq \bar{p}$ such that c is constant on $[\bar{q}]^2 \setminus \mathrm{diag}$?

'Yes', if c is *continuous* on $[\bar{p}]^2 \setminus \mathrm{diag}$ (Geschke-Kojman-Kubiś-Schipperus)

For more complicated c, there are combinatorial obstructions to such a straightforward generalization.

A generalization of Galvin's Theorem

For $\bar{x}_0, \bar{x}_1 \in [\bar{p}]$ (a subspace of $(2^\omega)^{\mathrm{supp}(\bar{p})}$), let

$$\Delta(\bar{x}_0, \bar{x}_1) = \text{the least } \xi \in \mathrm{supp}(\bar{p}) \text{ such that } \bar{x}_0(\xi) \neq \bar{x}_1(\xi).$$

Theorem (S. 2016)

For every $\bar{p} \in \mathbb{P}$ and every symmetric universally Baire

$$c \colon [\bar{p}]^2 \to \{0, 1\}$$

there is $\bar{q} \in \mathbb{P}$, $\bar{q} \leq \bar{p}$, with an enumeration $\langle \sigma_k \mid k \in \omega \rangle$ of $\mathrm{supp}(\bar{q})$ and a function $N \colon \mathrm{supp}(\bar{q}) \to \omega$ such that for $(\bar{x}_0, \bar{x}_1) \in [\bar{q}]^2 \setminus \mathrm{diag}$, the value of $c(\bar{x}_0, \bar{x}_1)$ only depends on $\Delta(\bar{x}_0, \bar{x}_1) = \xi$ and the following (finite) piece of information:

$$(\bar{x}_0 \restriction K, \bar{x}_1 \restriction K)$$

where $K = \{\sigma_0, \ldots, \sigma_{N(\xi)}\} \times N(\xi)$.

Above we simplify notation by identifying $(2^\omega)^{\mathrm{supp}(\bar{p})}$ and $2^{\omega \times \mathrm{supp}(\bar{p})}$.

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Outline

A basis for $\mathbb{R}$ over $\mathbb{Q}$

Hamel bases

Let $X = \mathbb{R}$ and let R_H be the set of finite tuples from X which are linearly dependent over $\mathbb{Q}$. An R_H-**mds** is usually known as a *Hamel basis*.

A more involved proof but using similar ideas as in the previous sketch gives us:

Theorem (S. 2016)

If s is a Sacks real over $\mathbf{L}$, there is a Π^1_1 Hamel basis in $\mathbf{L}[s]$.

In particular this uses another generalization of Galvin's theorem to k-tuples due to Blass.

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Outline

Ideas for further work

Conjecture

Every (not necessarily binary) Σ^1_1 relation has a Δ^1_2 maximal discrete set in the (iterated) Sacks extension of **L**.

Conjecture

There is a model where $2^\omega > \omega_1$ and *any* cofinitary group of size $< 2^\omega$ is a subgroup of a $\underline{\Pi}^1_2$ maximal cofinitary group.

Some open questions

1. (Mathias) Does "every projective set is completely Ramsey" imply "there is no projective mad family"?
2. Is there a Borel maximal incomparable set of Turing degrees?
3. (Horowitz-Shelah) Is there a Σ^1_1 relation R on a Polish space such that "there is no projective R-**mds**" is equiconsistent with, say, a measurable?

Thank You!

Large cardinals from mds

"there is no projective R-**mds**" is equiconsistent with ZFC in several other cases, as well:

- so-called *independent families of sets* (Brendle-Khomskii, unpublished)
- maximal orthogonal families of measures (Fischer-Törnquist, 2010); This is because "every projective set has the Baire property" $\Rightarrow$ "there are no projective maximal orthogonal families of measures", and the first statement is equiconsistent with ZFC.

The statement that there are no definable R-**mds** *can* have large cardinal strength:

Theorem (Horowitz-Shelah, 2016)

*There is a Borel binary relation R on 2^ω (in fact, a graph relation) such that "there is no projective R-**mds**" is equiconsistent with the existence of an inaccessible cardinal.*

It is not known whether one can obtain larger cardinals in this way.

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