Elementary embeddings and symmetric extensions
a study of critical cardinals

Joint work (in progress) with Yair Hayut

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This is not necessarily the case in ZF.

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This is a common "problem" when removing the axiom of choice from the equation: we lose the ability to translate between model theoretic and combinatorial properties.

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Theorem (Folklore)

If κ is a critical cardinal then κ is a regular limit cardinal, there is no $\alpha < \kappa$ such that there is a surjection from V_α onto κ; and there is a normal κ-complete (free) ultrafilter on κ.

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This allows for a model theoretic definition for many large cardinals whose combinatorial properties are inherently weaker without choice (e.g. strong cardinals).

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These cardinals play an important role in proofs related to the HOD Conjecture, and they are strong enough to reflect some non-trivial information about the universe.

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If κ is a supercompact cardinal, $\lambda < \kappa$ is a regular cardinal such that $\mathrm{DC}_{<\lambda}$ holds, then there is a forcing $\mathbb{P}^{\lambda}_{\kappa}$ such that if G is a V-generic filter for $\mathbb{P}^{\lambda}_{\kappa}$, then

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Corollary

If κ is a supercompact cardinal, then there is a forcing extension in which DC *holds.*

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Force with $\mathbb{P}^{\omega}_{\kappa}$, then $\kappa = \omega_1$, λ^+ is preserved, and DC holds. But DC implies that no successor cardinal has countable cofinality. So there is no short cofinal sequence in the forcing extension, and therefore there was no short cofinal sequence in V. □

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Successors of critical cardinals (cont.)

While we do not know the answer to any of the question above, we do know the following:

Theorem (Hayut–K.)

Assume ZFC, *and suppose that* κ *is measurable. Then there is a symmetric extension in which* κ *is a critical cardinal, and for some* $\lambda > \kappa$, $\mathrm{cf}(\lambda^+) = \omega$.

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Note that there is a problem with the proof without assuming some choice holds up to λ, since the collapse of λ^{+n} might add subsets to κ, and possibly destroying the fact that it is a critical cardinal.

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The proof is using a supercompact Radin forcing, and we can replace ω by any regular cardinal $\leq \kappa$.

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1. We are able to extend j to an elementary embedding from W to N.
2. This extension is sufficiently amenable to W, so W knows about an embedding witnessing that κ is critical. (In particular, W and N share an initial segment.)

Symmetric extensions

If $\mathbb{P}$ is a forcing, and π is an automorphism of $\mathbb{P}$, then π extends to $\mathbb{P}$-names via this recursive definition:

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- We say that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F}\rangle$ is a **symmetric system** if $\mathscr{G}$ is an automorphism group of $\mathbb{P}$ and $\mathscr{F}$ is a normal filter of subgroups of $\mathscr{G}$.

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Theorem (K.)

Assume V satisfies ZFC*, and let W be a symmetric extension using the system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$. Then V is definable in W, and the statement "I am a symmetric extension of V using the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$" is a first-order statement (with parameters from V) in the language of set theory.*

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It should be pointed, however, that the same symmetric extension can be obtained by wildly different symmetric systems.

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- We also fix some symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, and a V-generic filter G. We will use W to denote the symmetric extension these define. On the M side of things, we will use N to denote the symmetric extension of M obtained by $j(\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle)$.

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The most naive expectation would be that if j extends between $V[G]$ and $M[j(G)]$, then it can be extended between the symmetric submodels.

This is not always the case.

Example

Suppose that κ is a measurable cardinal immune under adding Cohen subsets. Consider the symmetric system $\langle \mathbb{P}, S_\kappa, \mathscr{F}_\kappa \rangle$, with:

1. $\mathbb{P} = \mathrm{Add}(\kappa, \kappa)$.
2. S_κ the group of permutations of κ, with $\pi p(\pi\alpha, \beta) = p(\alpha, \beta)$ for $p \in \mathbb{P}$.
3. $\mathscr{F}_\kappa$ the filter of subgroups generated by groups of the form $\mathrm{fix}(E) = \{\pi \in S_\kappa \mid \kappa \restriction E = \mathrm{id}\}$, for $E \in [\kappa]^{<\kappa}$.

The symmetric extension satisfies $\mathsf{DC}_{<\kappa}$, so if $j \colon V_{\kappa+1} \to N$ is any elementary embedding, $N \models \mathsf{DC}_{<j(\kappa)}$ and in particular N knows about a well-ordering of $V_{\kappa+1}$. But the symmetric extension does not know about such ordering.

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The same holds for symmetric extensions. As we shall see in a moment.

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Suppose that $\langle\mathbb{P},\mathcal{G},\mathcal{F}\rangle\in V_\kappa$. *Then* j *extends between* W *and* N, *and it is an amenable embedding.*

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The Levy–Solovay theorem can be exploited to obtain an amenable extension of the embedding between the symmetric extensions even if the embedding does not extend in the full generic extension. For example, intermediate models to adding a single Cohen real.

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We would like to get something similar in the context of symmetric extensions. But generic filters are not the correct objects to deal with in the case of symmetric extensions.

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2. *For every symmetrically generic G such that $p \in G$, $\mathsf{HS}^G \models \varphi(\dot{x}^G)$.*

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The proof is the same proof as in the ZFC case, utilizing the HS-forcing relation instead of the usual forcing relation. The extension of the embedding, however, is not necessarily amenable to W. But we can give some conditions under which the extended embedding is in fact amenable.

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If $(j(\pi)\dot{x})^H = \pi(\dot{x}^H)$, then $\left\{\left\langle \dot{x}, j(\dot{x})^H \right\rangle^\bullet \mid \dot{x} \in \mathsf{HS}\right\}^\bullet$ is stable under π. In particular, if all automorphisms in $\mathcal{G}$ satisfy this, the extension of the embedding is amenable.

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This is interesting, because up until now we had no examples where the extension of the embedding did not come from a Levy–Solovay type argument. And the fact that the failure happens all the way up to our critical point makes it more challenging to ensure that the embedding extends.

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What is the consistency strength of ZF $+ \kappa$ *supercompact* $+ \mathbb{P}^{\omega}_{\kappa}$ *does not force* AC?

Thank you for your attention!