Measuring together with the continuum large

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Definition

Measuring holds if and only if for every sequence $\vec{C} = (C_\delta : \delta \in \omega_1)$, if each C_δ is a closed subset of δ in the order topology, then there is a club $C \subseteq \omega_1$ such that for every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq C_\delta$, or
- $(C \setminus \alpha) \cap C_\delta = \emptyset$.

That is, a tail of $(C \cap \delta)$ is either contained in or disjoint from C_δ.

This principle is of course equivalent to its restriction to club-sequences $\vec{C}$ on ω_1.

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Measuring follows from BPFA and also from MRP.

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Theorem

(CH) Let κ be a cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$. There is then a partial order $\mathcal{P}$ with the following properties.

1. *$\mathcal{P}$ is proper.*
2. *$\mathcal{P}$ is $\aleph_2$–Knaster.*
3. *$\mathcal{P}$ forces measuring.*
4. *$\mathcal{P}$ forces $2^{\aleph_0} = 2^{\aleph_1} = \kappa$.*
5. *$\mathcal{P}$ forces $\mathfrak{b}(\omega_1) = cf(\kappa)$*

Recall that a poset is $\aleph_2$–Knaster iff every collection of $\aleph_2$–many conditions contains a subcollection of cardinality $\aleph_2$ consisting of pairwise compatible cond. Also, $\mathfrak{b}(\omega_1)$ denotes the minimal cardinality of an unbounded subset of ${}^{\omega_1}\omega_1$ mod. countable.

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The theorem will be proved by means of what can be described as a finite support iteration incorporating systems of ctble. struct. with symmetry requirements as side cond. In fact, our forcing $\mathcal{P}$ will be $\mathcal{P}_\kappa$, where $\mathcal{P}_\kappa$ is the last step of this iteration. The actual construction is a variation of previous works.

There are 2 main new ingredients in our present construction. Specifically, at any given stage $\beta < \kappa$ of the iteration,

(a) the set $\mathcal{N}^q_\beta$ of models N that are active at that stage, in the sense that $\beta \in N$ and that the marker associated to N at that stage is β, is actually a T–symmetric system (for a suitable predicate T), and

(b) if $\beta = \alpha + 1$, we use a separate symmetric system in the working part at α included in the above symmetric system corresponding to the previous stage, i.e., in $\mathcal{N}^q_\alpha$; these are the symmetric systems we will denote by $\mathcal{O}_{q,\alpha}$.

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This use of local symmetry is crucial in the verification that measuring holds in the final generic extension. Specifically, it is needed in the verification that the generic club C added at a stage α will be such that for every $\delta \in Lim(\omega_1)$, a tail of $C \cap \delta$ will be contained in C_δ in case we could not make the promise of avoiding C_δ (where C_δ is the δ–indexed member of the club–sequence picked at stage α).

In a paper from the 80's, Abraham and Shelah build, given any cardinal $\lambda \geq \aleph_2$, a forcing notion $\mathcal{P}$ which, if CH holds, preserves cardinals and is such that if G is $\mathcal{P}$–generic over V, then in $V[G]$ there is a family $\mathcal{C}$ of size λ consisting of clubs of ω_1 and with the property that, in any outer model M of $V[G]$ with the same ω_1 and ω_2 as $V[G]$, there is no club E of ω_1 in M diagonalising $\mathcal{C}$ (where E diagonalising $\mathcal{C}$ means that $E \setminus D$ is bounded in ω_1 for each $D \in \mathcal{C}$).

CH necessarily fails in the Abraham–Shelah model $V[G]$ since, by a result of Galvin, CH implies that for every family $\mathcal{C}$ of size $\aleph_2$ consisting of clubs of ω_1 there is an uncountable $\mathcal{C}' \subseteq \mathcal{C}$ such that $\bigcap \mathcal{C}'$ is a club.

It is not difficult to see that the generic club added at every stage $\alpha < \kappa$ of our iteration diagonalises all clubs of ω_1 from $\mathbb{V}[G_\alpha]$ (where G_α is the generic filter at that stage). So, it would be impossible to run anything like our iteration over the Abraham–Shelah model without collapsing ω_2, and therefore we should start from a ground model which is sufficiently different from the Abraham–Shelah model. That is accomplished by imposing that *CH* must be true in our ground model.

Question: Is it consistent to have measuring together with $\mathfrak{b}(\omega_1) = \aleph_2$ and $2^{\aleph_1} > \aleph_2$?.

Important problem: Is measuring compatible with CH?

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Notation. if $N \cap \omega_1 \in \omega_1$, then $\delta_N := N \cap \omega_1$.

Definition

Let $T \subseteq H(\theta)$ and let $\mathcal{N}$ be a finite set of countable subsets of $H(\theta)$. We will say that $\mathcal{N}$ *is a* T*–symmetric system* iff

(A) For every $N \in \mathcal{N}$, $(N, \in, T) \prec (H(\theta), \in, T)$.

(B) Given distinct N, N' in $\mathcal{N}$, if $\delta_N = \delta_{N'}$, then there is a unique isomorphism

$$\Psi_{N,N'} : (N, \in, T) \longrightarrow (N', \in, T)$$

Furthermore, $\Psi_{N,N'}$ is the identity on $N \cap N'$.

(C) $\mathcal{N}$ is closed under isomorphisms. That is, for all N, N', M in $\mathcal{N}$, if $M \in N$ and $\delta_N = \delta_{N'}$, then $\Psi_{N,N'}(M) \in \mathcal{N}$.

(D) For all N, M in $\mathcal{N}$, if $\delta_M < \delta_N$, then there is some $N' \in \mathcal{N}$ such that $\delta_{N'} = \delta_N$ and $M \in N'$.

Remark. In all practical cases $\bigcup T = H(\theta)$, so T does determine $H(\theta)$ in these cases.

The following lemmas are proved in TAMS, vol. 367 (2015), 6103-6129.

Lemma

Let $T \subseteq H(\theta)$ and let N and N' be countable elementary substructures of $(H(\theta), \in, T)$. Suppose $\mathcal{N} \in N$ is a T–symmetric system and $\Psi : (N, \in, T) \longrightarrow (N', \in, T)$ is an isomorphism. Then $\Psi(\mathcal{N}) = \Psi``\mathcal{N}$ is also a T–symmetric system.

Lemma

Let $T \subseteq H(\theta)$, let $\mathcal{N}$ be a partial T–symmetric system and let $N \in \mathcal{N}$. Then the following holds.

1. *$\mathcal{N} \cap N$ is a T–symmetric system.*
2. *Suppose $\mathcal{N}^* \in N$ is a T–symmetric system such that $\mathcal{N} \cap N \subseteq \mathcal{N}^*$. Let*

$$\mathcal{M} = \mathcal{N} \cup \bigcup \{\Psi_{N,N'} \text{“} \mathcal{N}^* \,:\, N' \in \mathcal{N},\, \delta_{N'} = \delta_N\}$$

   Then $\mathcal{M}$ is the $\subseteq$–minimal T–symmetric system $\mathcal{W}$ such that $\mathcal{N} \cup \mathcal{N}^* \subseteq \mathcal{W}$.

Given $T \subseteq H(\theta)$ and T–symmetric systems $\mathcal{N}_0$, $\mathcal{N}_1$, let us write $\mathcal{N}_0 \cong \mathcal{N}_1$ iff

- $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1) = R$ and
- for some $m < \omega$, there are enumerations $(N_i^0)_{i<m}$ and $(N_i^1)_{i<m}$ of $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively, together with an isomorphism between

$$\langle \bigcup \mathcal{N}_0, \in, T, R, N_i^0 \rangle_{i<m}$$

and

$$\langle \bigcup \mathcal{N}_1, \in, T, R, N_i^1 \rangle_{i<m}$$

which is the identity on R.

Lemma

Let $T \subseteq H(\theta)$ and let $\mathcal{N}_0$ and $\mathcal{N}_1$ be T–symmetric systems. Suppose $\mathcal{N}_0 \cong \mathcal{N}_1$. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a T–symmetric system.

3 More preparations

1 Given sets N, $\mathcal{X}$ and an ordinal η, we define Rank $(\mathcal{X}, N) \geq \eta$ recursively by:

- Rank $(\mathcal{X}, N) \geq 1$ if and only if for every $a \in N$ there is some $M \in \mathcal{X} \cap N$ such that $a \in M$.
- If $\eta > 1$, then Rank $(\mathcal{X}, N) \geq \eta$ if and only if for every $\eta' < \eta$ and every $a \in N$ there is some $M \in \mathcal{X} \cap N$ such that $a \in M$ and Rank $(\mathcal{X}, M) \geq \eta'$.

2 Now let $\Phi : \kappa \longrightarrow H(\kappa)$ be such that $\Phi^{-1}(x)$ is unbounded in κ for all $x \in H(\kappa)$. Let also $\lhd$ be a well–order of $H((2^{\kappa})^{+})$. Notice that Φ exists by $2^{<\kappa} = \kappa$.

3 Let $(\theta_\alpha)_{\alpha<\kappa}$ be the seq. of card. defined by $\theta_0 = |H((2^\kappa)^+)|^+$ and $\theta_\alpha = (2^{\sup_{\beta<\alpha}\theta_\beta})^+$ if $\alpha > 0$. For each $\alpha < \kappa$ let $\mathcal{M}^*_\alpha$ be the collection of all ctble. el. substruct. of $H(\theta_\alpha)$ containing Φ, $\lhd$ and $(\theta_\beta)_{\beta<\alpha}$, and let $\mathcal{M}_\alpha = \{N^* \cap H(\kappa) : N^* \in \mathcal{M}^*_\alpha\}$. Let T^α be the $\lhd$–first $T \subseteq H(\kappa)$ such that for every $N \in [H(\kappa)]^{\aleph_0}$, if $(N, \in, T \cap N) \prec (H(\kappa), \in, T)$, then $N \in \mathcal{M}_\alpha$. Let also

$$\mathcal{T}^\alpha = \{N \in [H(\kappa)]^{\aleph_0} : (N, \in, T^\alpha \cap N) \prec (H(\kappa), \in, T^\alpha)\}.$$

So, $\mathcal{T}^\alpha$ is a club of el. substruct. of $H(\kappa)$. Its elements are coded by a certain pred. T^α and they are project. of some other el. substruct. of $H(\theta_\alpha)$, where $(\theta_\alpha)_{\alpha<\kappa}$ is a canonical seq.

Fact. Let $\alpha < \beta \leq \kappa$.

1. If $N^* \in \mathcal{M}^*_\beta$ and $\alpha \in N^*$, then $\mathcal{M}^*_\alpha \in N^*$ and $N^* \cap H(\kappa) \in \mathcal{T}^\alpha$.
2. If $N, N' \in \mathcal{T}^\beta$, $\Psi : (N, \in, T^\beta \cap N) \longrightarrow (N', \in, T^\beta \cap N')$ is an isomorphism, and $M \in N \cap \mathcal{T}^\beta$, then $\Psi(M) \in \mathcal{T}^\beta$.

The forcing construction

Our forcing $\mathcal{P}$ will be $\mathcal{P}_\kappa$, where $(\mathcal{P}_\beta : \beta \leq \kappa)$ is the sequence of posets to be defined next.
From now on, if q is an ordered pair (F, Δ), we will denote F and Δ by F_q and Δ_q, respectively.
Let $\beta \leq \kappa$ and suppose $\mathcal{P}_\alpha$ has been defined for all $\alpha < \beta$.
Conditions in $\mathcal{P}_\beta$ are ordered pairs $q = (F, \Delta)$ satisfying:

1. F is a finite function with $\text{dom}\,(F) \subseteq \beta$.
2. Δ is a finite set of pairs (N, γ) such that $N \in [H(\kappa)]^{\aleph_0}$ and γ is an ordinal such that $\gamma \leq \beta$ and $\gamma \leq \sup(N \cap \kappa)$.
3. $\mathcal{N}^q_\beta := \{N : (N, \beta) \in \Delta,\ \beta \in N\}$ is a T^β–symmetric system.
4. For every $\alpha < \beta$, the restriction of q to α,
$$q|_\alpha := (F \restriction \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\}),$$
is a condition in $\mathcal{P}_\alpha$.

(5) Suppose $\beta = \alpha + 1$. Let $\mathcal{N}^{\dot{G}_\alpha}$ be a $\mathcal{P}_\alpha$–name for

$$\bigcup\{\mathcal{N}_\alpha^r \ : \ r \in \dot{G}_\alpha\}.$$

Assume that $\dot{C}^\alpha := \Phi(\alpha)$ is a $\mathcal{P}_\alpha$–name for a club–seq. on ω_1. If $\alpha \in \mathrm{dom}\,(F)$, then $F(\alpha) = (p, b, \mathcal{O})$ has the following prop.

(a) $p \subseteq \omega_1 \times \omega_1$ is a finite strictly increasing function.

(b) $\mathcal{O} \subseteq \mathcal{N}_\alpha^{q|\alpha}$ is a T^β–symmetric system.

(c) $\mathrm{ran}\,(p) \subseteq \{\delta_N \ : \ N \in \mathcal{O}\}$

(d) For every $\delta \in \mathrm{dom}\,(p)$, if $N \in \mathcal{O}$ is such that $p(\delta) = \delta_N$, then

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{Rank}\,(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^\beta, N) \geq \delta$$

(e) $\mathrm{dom}\,(b) \subseteq \mathrm{dom}\,(p)$ and $b(\delta) < p(\delta)$ for every $\delta \in \mathrm{dom}\,(b)$.

(f) For every $\delta \in \mathrm{dom}\,(b)$,

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{ran}\,(p \restriction \delta) \cap \dot{C}^\alpha(p(\delta)) \subseteq b(\delta)$$

(g) For every $\delta \in \mathrm{dom}\,(b)$, if $N \in \mathcal{O}$ is such that $p(\delta) = \delta_N$, then

$$q|_\alpha \Vdash_{\mathcal{P}_\alpha} \mathrm{Rank}\,(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^\beta \,:\, \delta_M \notin \dot{C}^\alpha(p(\delta))\}, N) \geq \delta$$

(h) If $N \in \mathcal{N}^q_\beta$, then $N \in \mathcal{O}$, $\delta_N \in \mathrm{dom}\,(p)$ and $p(\delta_N) = \delta_N$.

Given $\mathcal{P}_\beta$–conditions $q_i = (F_i, \Delta_i)$, for $i = 0$, 1, q_1 extends q_0 iff

- dom $(F_0) \subseteq$ dom (F_1) and for all $\alpha \in$ dom (F_0), if $F_0(\alpha) = (p, b, \mathcal{O})$ and $F_1(\alpha) = (p', b', \mathcal{O}')$, then $p \subseteq p'$, $b \subseteq b'$ and $\mathcal{O} \subseteq \mathcal{O}'$, and
- for every $(N, \gamma) \in \Delta_0$ there is some $\gamma' \geq \gamma$ such that $(N, \gamma') \in \Delta_1$.

If $q \in \mathcal{P}_\beta$ for some $\beta \leq \kappa$, we will use $\mathrm{supp}\,(q)$ to denote the domain of F_q ($\mathrm{supp}\,(q)$ stands for the support of q). Also, if $\alpha \in \mathrm{supp}\,(q)$ and $F_q(\alpha) = (p, b, \mathcal{O})$, then $p_{q,\alpha}$, $b_{q,\alpha}$ and $\mathcal{O}_{q,\alpha}$ denote p, b and $\mathcal{O}$, respectively.

Lemma

Let $\alpha \leq \beta \leq \kappa$. If $q = (F_q, \Delta_q) \in \mathcal{P}_\alpha$, $r = (F_r, \Delta_r) \in \mathcal{P}_\beta$, and $q \leq_\alpha r|_\alpha$, then

$$r \wedge_\alpha q := (F_q \cup (F_r \restriction [\alpha, \beta)), \Delta_q \cup \Delta_r)$$

is a condition in $\mathcal{P}_\beta$ extending r. Hence, $\mathcal{P}_\alpha$ is a complete suborder of $\mathcal{P}_\beta$.

Proof. The proof depends on the use of the markers in the definition of the forcing. The fact that a marker γ is associated to a submodel N in a condition (F, Δ) (i.e., the fact that $(N, \gamma) \in \Delta$) tells us that N is 'active', for that condition, up to and including stage γ in the iteration. New side conditions (N, γ) appearing in Δ_q may well be such that $N \cap [\alpha, \beta)$ is nonempty, but they will not impose any problematic promises on ordinals occurring in the interval $[\alpha, \beta)$ simply because $\gamma \leq \alpha$.

Lemma

For every ordinal $\alpha \le \kappa$, $\mathcal{P}_\alpha$ is $\aleph_2$–Knaster.

Proof. Case $\alpha = 0$. Suppose $m < \omega$ and $q_\xi = \{N_i^\xi \,:\, i < m\}$ is a $\mathcal{P}_0$–condition for each $\xi < \omega_2$. Of course, we are identifying a $\mathcal{P}_0$–condition q with dom (Δ_q). By CH we may assume that

$$\{\bigcup_{i<m} N_i^\xi \,:\, \xi < \omega_2\}$$

forms a Δ–system with root X. Furthermore, again by CH, we may assume that, for all ξ, $\xi' < \omega_2$, the structures

$$\langle \bigcup_{i<m} N_i^\xi, \in, X, T^0, N_i^\xi \rangle_{i<m} \text{ and } \langle \bigcup_{i<m} N_i^{\xi'}, \in, X, T^0, N_i^{\xi'} \rangle_{i<m}$$

are isomorphic and that the corresponding iso. fixes X. This is true since there are only $\aleph_1$–many isomorphism types for such structures and since the only isomorphism between X and itself is the identity. So, $q_\xi \cup q_{\xi'}$ extends both q_ξ and $q_{\xi'}$.

For general α, suppose that q_ξ is a $\mathcal{P}_\alpha$–condition for each $\xi < \omega_2$. We may assume that there is some $m < \omega$ such that we may write

$$\mathrm{dom}\,(\Delta_{q_\xi}) = \{N_i^\xi \,:\, i < m\}$$

for each ξ. Let

$$\vec{T} = \{(a, \gamma) \,:\, \gamma \leq \alpha,\ a \in T^\gamma\}$$

By an argument as in the case $\alpha = 0$, we are allowed to adopt the point of view that $\{\bigcup_{i<m} N_i^\xi \,:\, \xi < \omega_2\}$, for $\xi < \omega_2$, forms a Δ–system with root X and that for all ξ, $\xi' < \omega_2$, the structures

$$\langle \bigcup_{i<m} N_i^\xi, \in, X, \vec{T}, N_i^\xi \rangle_{i<m} \text{ and } \langle \bigcup_{i<m} N_i^{\xi'}, \in, X, \vec{T}, N_i^{\xi'} \rangle_{i<m}$$

are isomorphic.

We may also assume that there is a finite set $x \subseteq X$ such that

- $\{\text{supp}\,(q_\xi) : \xi \in \omega_2\}$ forms a Δ–system with root x, and
- for all $\xi, \xi' \in \omega_2$ and for all $\alpha \in x$, $(p_{q_\xi,\alpha}, b_{q_\xi,\alpha}) = (p_{q_{\xi'},\alpha}, b_{q_{\xi'},\alpha})$.

Finally, again by the same argument as above, we may assume that for all ξ, $\xi' \in \omega_2$ and all $\gamma \in x$, $\mathcal{O}_{\xi,\gamma} \cup \mathcal{O}_{\xi',\gamma}$ is a T^γ–symmetric system. So, for all ξ, ξ', $(F_{q_\xi} \cup F_{q_{\xi'}}, \Delta_{q_\xi} \cup \Delta_{q_{\xi'}})$ is a condition in $\mathcal{P}_\alpha$ witnessing the compatibility of q_ξ and $q_{\xi'}$.

Lemma

$\mathcal{P}_\kappa$ *forces measuring.*

Proof. Let $\alpha < \kappa$, let G be $\mathcal{P}_\alpha$–generic, and suppose $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$–name for a club–sequence on ω_1. Let $\vec{C} = \Phi(\alpha)_G = (C_\epsilon \,:\, \epsilon \in Lim(\omega_1))$. Let H be a $\mathcal{P}_{\alpha+1}$–generic filter such that $H \restriction \mathcal{P}_\alpha = G$, and let $C = \bigcup \mathrm{ran}\,\{p_{q,\alpha} \,:\, q \in H\}$. By the $\aleph_2$–c.c. of $\mathcal{P}_\kappa$ and the choice of Φ, the conclusion will follow if we show that C is a club of ω_1 measuring $\vec{C}$.

By condition (5) (d) in the def. of our iteration it follows that C is a club of ω_1. Also, if $\epsilon \in C$ is such that there is some $q \in H$ such that $\epsilon = p_{q,\alpha}(\delta)$ for some $\delta \in \mathrm{dom}\,(b_{q,\alpha})$, then a tail of $C \cap \epsilon$ is disjoint from C_ϵ (by (5) (e), (f) in the def. of the iteration).

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Hence, it suffices to show that if $\delta \in \omega_1$ is such that $\delta \notin \mathrm{dom}\,(b_{q,\alpha})$ for every $q \in H$ and ϵ is such that $p_{q,\alpha}(\delta) = \epsilon$ for some $q \in H$, then a tail of $C \cap \epsilon$ is contained in C_ϵ.

But this implies, by (5) (g) and the usual density argument, that there is some $q \in H$ and some $N \in \mathcal{O}_{q,\alpha}$ such that $p_{q,\alpha}(\delta) = \delta_N$ and such that $q|_\alpha$ forces, in $\mathcal{P}_\alpha$, that $\mathrm{Rank}\,(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1} \,:\, \delta_M \notin \Phi(\alpha)(\epsilon)\}, N) = \delta_0$ for some given $\delta_0 < \delta$.

It will now be enough to find some $\eta \in [\delta_0,\, \delta)$ and some extension q^* of q such that every extension q' of q^* is such that $q'|_\alpha$ forces that $p_{q',\alpha}(\delta') \in \Phi(\alpha)(\delta)$ for every $\delta' \in \mathrm{dom}\,(p_{q',\alpha}) \cap [\eta,\, \delta)$.

Hence, it suffices to show that if $\delta \in \omega_1$ is such that $\delta \notin \text{dom}\,(b_{q,\alpha})$ for every $q \in H$ and ϵ is such that $p_{q,\alpha}(\delta) = \epsilon$ for some $q \in H$, then a tail of $C \cap \epsilon$ is contained in C_ϵ.

But this implies, by (5) (g) and the usual density argument, that there is some $q \in H$ and some $N \in \mathcal{O}_{q,\alpha}$ such that $p_{q,\alpha}(\delta) = \delta_N$ and such that $q|_\alpha$ forces, in $\mathcal{P}_\alpha$, that $\text{Rank}\,(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1} : \delta_M \notin \Phi(\alpha)(\epsilon)\}, N) = \delta_0$ for some given $\delta_0 < \delta$.

It will now be enough to find some $\eta \in [\delta_0, \delta)$ and some extension q^* of q such that every extension q' of q^* is such that $q'|_\alpha$ forces that $p_{q',\alpha}(\delta') \in \Phi(\alpha)(\delta)$ for every $\delta' \in \text{dom}\,(p_{q',\alpha}) \cap [\eta, \delta)$.

Hence, it suffices to show that if $\delta \in \omega_1$ is such that $\delta \notin \text{dom}\,(b_{q,\alpha})$ for every $q \in H$ and ϵ is such that $p_{q,\alpha}(\delta) = \epsilon$ for some $q \in H$, then a tail of $C \cap \epsilon$ is contained in C_ϵ.

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Claim. By extending $q|_\alpha$ if necessary we may assume that there is some $a \in N$ such that $q|_\alpha$ forces that if $M \in N \cap \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}$ is such that $a \in M$ and $\text{Rank}\,(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) \geq \delta_0$, then $\delta_M \in \Phi(\alpha)(\epsilon)$.

Proof of the claim. Let us work in $\mathbb{V}^{\mathcal{P}_\alpha \restriction q|_\alpha}$. If the conclusion fails, then for every $a \in N$ there is some $M \in N \cap \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}$ such that $a \in M$, $\delta_M \notin \Phi(\alpha)(\epsilon)$ and $\text{Rank}\,(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) \geq \delta_0$. Fix any such M. By the openness of $\epsilon \setminus \Phi(\alpha)(\epsilon)$ there is some $\rho < \delta_M$ such that $[\rho, \delta_M) \cap \Phi(\alpha)(\epsilon) = \emptyset$.

Now, if Rank $(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) = \delta^*$, then for every $\gamma < \delta^*$ and every $b \in M$ there is some $M' \in M \cap \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}$ such that $\{b, \rho\} \in M'$ and Rank $(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M') \geq \gamma$, and of course $\delta_{M'} \notin \Phi(\alpha)(\epsilon)$ by the above choice of ρ since $\delta_{M'} \in [\rho, \delta_M)$.

Iterating this argument we then have that Rank $(\{M' \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1} : \delta_{M'} \notin \Phi(\alpha)(\epsilon)\}, M) = \delta^*$. This shows that Rank $(\{M \in \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1} : \delta_{M'} \notin \Phi(\alpha)(\epsilon)\}, N) > \delta_0$ since $\delta_M \notin \Phi(\alpha)(\epsilon)$, which is a contradiction. This ends the proof of the claim.

Now, if Rank $(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) = \delta^*$, then for every $\gamma < \delta^*$ and every $b \in M$ there is some $M' \in M \cap \mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}$ such that $\{b, \rho\} \in M'$ and Rank $(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M') \geq \gamma$, and of course $\delta_{M'} \notin \Phi(\alpha)(\epsilon)$ by the above choice of ρ since $\delta_{M'} \in [\rho, \delta_M)$.

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Again by extending $q|_\alpha$ if necessary, we may also assume that there is some $M \in N \cap \mathcal{N}_\alpha^{q|_\alpha} \cap \mathcal{T}^{\alpha+1}$ containing all relevant objects, where this includes a, and such that $q|_\alpha$ forces Rank $(\mathcal{N}^{\dot{G}_\alpha} \cap \mathcal{T}^{\alpha+1}, M) = \delta_1$, where $\delta_1 < \delta$ is such that $\delta_1 > \max(\mathrm{dom}\,(p_{q,\alpha} \restriction \delta))$ and $\delta_1 \geq \delta_0$.

Let now q^* be any ext. of q such that $M \in \mathcal{O}_{q^*,\alpha}$ and such that $p_{q^*,\alpha}(\delta_1) = \delta_M$. We claim that $\eta = \delta_1$ and q^* are as desired.

Indeed, it suffices to note that if q' is any cond. extending q^* and $R \in \mathcal{O}_{q',\alpha}$ is such that $\delta_R > \delta_M$ and $\delta_R < \delta_N$, then $q'|_\alpha \Vdash_{\mathcal{P}_\alpha} \delta_R \in \Phi(\alpha)(\epsilon)$. But by symmetry of $\mathcal{O}_{q',\alpha}$ there is some $R' \in \mathcal{O}_{q',\alpha} \cap N$ such that $M \in R'$ and $\delta_{R'} = \delta_R$. Since $a \in R'$ and $q'|_\alpha$ extends $q^*|_\alpha$, it follows then that

$$q'|_\alpha \Vdash_{\mathcal{P}_\alpha} \delta_R = \delta_{R'} \in \Phi(\alpha)(\epsilon).$$

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$$q'|_\alpha \Vdash_{\mathcal{P}_\alpha} \delta_R = \delta_{R'} \in \Phi(\alpha)(\epsilon).$$

A similar argument (using properness) shows that if H is a $\mathcal{P}_{\alpha+1}$–generic filter and $C = \bigcup \mathrm{ran}\,\{f_{q,\alpha} : q \in H\}$, then C diagonalises all clubs of ω_1 in $V[G]$, where $G = H \cap \mathcal{P}_\alpha$. By the $\aleph_2$–c.c. of $\mathcal{P}_\kappa$, it follows that $\mathcal{P}_\kappa$ forces $\mathfrak{b}(\omega_1) = cf(\kappa)$

Properness

Given $\alpha < \kappa$, a condition $q \in \mathcal{P}_\alpha$, and a countable elementary substructure N of $H(\kappa)$, we will say that q *is* $(N, \mathcal{P}_\alpha)$*–pre-generic* in case $(N, \alpha) \in \Delta_q$.

Also, given a countable elementary substructure N of $H(\kappa)$ and a $\mathcal{P}_\alpha$–condition q, we will say that q *is* $(N, \mathcal{P}_\alpha)$*–generic* iff q forces $\dot{G}_\alpha \cap A \cap N \neq \emptyset$ for every maximal antichain A of $\mathcal{P}_\alpha$ such that $A \in N$.

Note that this is more general than the standard notion of $(N, \mathbb{P})$–genericity, for a forcing notion $\mathbb{P}$, which applies only if $\mathbb{P} \in N$. Indeed, in our situation $\mathcal{P}_\alpha$ is of course never a member of N if $N \subseteq H(\kappa)$.

Lemma

Suppose $\alpha < \kappa$ and $N \in \mathcal{T}^{\alpha+1}$. Then the following holds.

$(1)_\alpha$ *For every $q \in N$ there is $q' \leq_\alpha q$ such that q' is $(N, \mathcal{P}_\alpha)$–pre-generic.*

$(2)_\alpha$ *If $q \in \mathcal{P}_\alpha$ is $(N, \mathcal{P}_\alpha)$–pre-generic, then q is $(N, \mathcal{P}_\alpha)$–generic.*

Instances of the inductive proof on α. The case $\alpha = 0$ is well known and so, we omit it. Let us proceed to the case $(1)_\alpha$ with $\alpha = \sigma + 1$. By $(1)_\sigma$ we may assume, by extending $q|_\sigma$, that $q|_\sigma$ is $(N, \mathcal{P}_\sigma)$–pre-generic. So, if $\sigma \notin \operatorname{supp}(q)$, then $q' = (F_q, \Delta_q \cup \{(N, \alpha)\}$ witnesses $(1)_\alpha$.

Assume that $\sigma \in \operatorname{supp}(q)$. Since $q|_\sigma$ is $(N, \mathcal{P}_\sigma)$–pre-gen., $q|_\sigma$ forces in $\mathcal{P}_\sigma$ that $N \in \mathcal{N}^{\dot{G}_\sigma}$. So, $q|_\sigma$ forces that for every $x \in N$ there is $M \in \mathcal{N}^{\dot{G}^\sigma} \cap \mathcal{T}^{\sigma+1}$ such that $x \in M$ (as witnessed by N).

Let us work in $V^{\mathcal{P}_\sigma \restriction q|_\sigma}$. Since

$$\langle N[\dot{G}_\sigma], \dot{G}_\sigma, T^{\sigma+2}, H(\kappa)^V\rangle \preccurlyeq \langle H(\kappa)[\dot{G}_\sigma], \dot{G}_\sigma, T^{\sigma+2}, H(\kappa)^V\rangle,$$

there is an M as above in $N[\dot{G}_\sigma] \cap V$ (where V denotes the ground model). We may also assume that $M \in N$, since $N[\dot{G}_\sigma] \cap V = N$ (which follows from $(2)_\sigma$ applied to N and $q|_\sigma$).

This shows that $q|_\sigma$ forces Rank $(\mathcal{N}^{\dot{G}_\sigma} \cap \mathcal{T}^{\sigma+1}, N) \geq 1$. In fact, by iterating this argument we can show, by induction on μ, that $q|_\sigma$ forces Rank $(\mathcal{N}^{\dot{G}_\sigma} \cap \mathcal{T}^{\sigma+1}, N) \geq \mu$ for every $\mu < \delta_N$. In view of these considerations, it suffices to define q' as the condition $(F', \Delta_q \cup \{(N, \alpha)\})$, where F' extends F_q and

$$F'(\sigma) = (p_{q,\sigma} \cup \{\langle \delta_N, \delta_N\rangle\}, b_{q,\sigma}, \mathcal{O}_{q,\sigma} \cup \{N\})$$

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