Ordinal definable subsets of singular cardinals

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Arctic Set Theory

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For example: the powerset function for singular cardinals.

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1. Silver: SCH cannot fail for the first time at a singular cardinal with uncountable cofinality.
2. Magidor: SCH can consistently fail at $\aleph_\omega$, from large cardinals.

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- We show that this is not the case for countable cofinalities. More precisely, we construct a forcing extension where the above fails.

The main result

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Note: work during a SQuaRE at AIM; thank you AIM!

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    each $x_n^\alpha \in \mathcal{P}_{\kappa_n}(\alpha)$, and $\alpha = \cup_n x_\alpha$.
- Preserves cardinals up to κ and above λ and makes $\lambda = \kappa^+$.

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- $M \subset M[G]$,
- $G \in M[G]$ and $G \notin M$,
- information about $M[G]$ can be obtained while working in M.

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  Cardinals are preserved, due to the *Prikry property*.

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- In our construction: $\lambda > \kappa = \sup_n \kappa_n$. Use many supercompact measures to add ω sequences through $\mathcal{P}_{\kappa_n}(\alpha)$ for unboundedly many $\alpha < \lambda$.

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- We will use the sequence of measures $\langle U_{n,\alpha} \mid n < \omega \rangle$ to add $\langle x_n^\alpha \mid n < \omega \rangle$.
- The support of the conditions is quite large, which is necessary for preservation of cardinals.

The forcing

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Conditions in $\mathbb{P}$ are of the form:

$$p = \langle f_0, ..., f_{n-1}, \langle a_n, A_n, f_n \rangle, \langle a_{n+1}, A_{n+1}, f_{n+1} \rangle, ... \rangle,$$

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- $\leq^*$ restricted to conditions of length n is κ_n-closed.

The last two properties give:

1. no new bounded subsets of κ;
2. preservation of λ.

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Actually, with some more work and stronger assumptions, can make λ supercompact in $HOD_x^{V[G]}$.