Infinitely often equal trees and Cohen reals

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DER FORSCHUNG | DER LEHRE | DER BILDUNG

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$A \subseteq \omega^\omega$ is a **countably infinitely often equal (ioe) family** iff

$$\forall \{x_i \mid i < \omega\} \, \exists y \in A : \;\; y \text{ is ioe to every } x_n.$$

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Definition

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If T is a full-splitting Miller tree then $[T]$ is a countably ioe family (does everyone agree?)

Perfect-set-type theorem

Theorem (Spinas 2008)

Every analytic countably ioe family contains $[T]$ for some full-splitting Miller tree T.

Otmar Spinas, *Perfect set theorems*, Fundamenta Mathematicae 201 (2): 179–195, 2008.

Idealized Forcing

We were mainly interested in Spinas' result because of "Idealized Forcing"

- Let $\mathfrak{I}_{\text{ioe}} := \{A \subseteq \omega^\omega \mid A \text{ is } \textbf{not} \text{ a countably ioe family.}\}$
- Then $\text{Borel}(\omega^\omega)/\mathfrak{I}_{\text{ioe}}$ is a forcing for generically adding an **ioe real** (i.e., a real which is ioe to all ground model reals).
- By the dichotomy of Spinas:

$$\mathbb{FM} \hookrightarrow_d \text{Borel}(\omega^\omega)/\mathfrak{I}_{\text{ioe}}.$$

where $\mathbb{FM}$ denotes the collection of full-splitting Miller trees.

What happened

Giorgio and I began working on some questions about this forcing ...

... and we obtained contradictory results!

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Spinas' Dichotomy Theorem

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Every analytic countably ioe family contains $[T]$ *for some full-splitting Miller tree* T.

Counterexample

Let T be the tree on $\omega^{<\omega}$ defined as follows:

- If $|s|$ is even then $\mathrm{succ}_T(s) = \{0, 1\}$.
- If $|s|$ is odd then $\mathrm{succ}_T(s) = \begin{cases} 2\mathbb{N} & \text{if} \quad s(|s|-1) = 0 \\ 2\mathbb{N}+1 & \text{if} \quad s(|s|-1) = 1 \end{cases}$

Then $[T]$ is a countably ioe family not containing a full-splitting Miller subtree.

New tree

Definition (Spinas)

A tree $T \subseteq \omega^\omega$ is called an **infinitely often equal tree (ioe-tree)**, if for each $t \in T$ there exists $N > |t|$, such that for every $k \in \omega$ there exists $s \in T$ extending t such that $s(N) = k$.

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Theorem (Spinas 2008)

Every analytic countably ioe family contains $[T]$ for some **ioe-tree** T.

Let $\mathbb{IE}$ denote the partial order of ioe-trees, ordered by inclusion:

$$\mathbb{IE} \hookrightarrow_d \mathrm{Borel}(\omega^\omega)/\mathfrak{I}_{\mathrm{ioe}}$$

Cohen reals

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Question

Does $\mathbb{IE}$ add Cohen reals?

Half a Cohen real

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Question (Fremlin)

Is there a forcing adding $\frac{1}{2}$Cohen real without adding a Cohen real?

Zapletal's soluton

Theorem (Zapletal 2013)

Let X be a compact metrizable space which is infinite-dimensional, and all of its compact subsets are either infinite-dimensional or zero-dimensional. Let $\mathfrak{I}$ be the σ-ideal σ-generated by the compact zero-dimensional subsets of X. Then $\mathrm{Borel}(X)/\mathfrak{I}$ adds an ioe real but not a Cohen real.

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Definition

A forcing $\mathbb{P}$ has the **meager image property (MIP)** iff for every continuous $f : \omega^\omega \to \omega^\omega$ there exists $T \in \mathbb{P}$ such that $f``[T]$ is meager.

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How is this related to not adding Cohen reals?

If we could prove the MIP **below an arbitrary condition** $S \in \mathbb{E}$, then we would know that $\mathbb{E}$ does not add Cohen reals.

Why? Using **continuous reading of names**, for every name for a real $\dot{x}$ there is $S \in \mathbb{E}$ and continuous $f : [S] \to \omega^\omega$ such that $S \Vdash \dot{x} = f(\dot{x}_G)$. If $T \leq S$ is such that $f``[T] \in \mathcal{M}$ then $T \Vdash$ "$\dot{x} \in f``[T] \in \mathcal{M}$" and hence $T \Vdash$ "$\dot{x}$ is not Cohen".

Meager image property

Theorem (Kh-Laguzzi)

$\mathbb{IE}$ *has the MIP.*

Meager image property

Theorem (Kh-Laguzzi)

$\mathbb{E}$ *has the MIP.*

The proof of this theorem is weird:

Lemma

If $\mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ *then* $\mathbb{E}$ *has the MIP.*

Corollary

$\mathbb{E}$ *has the MIP.*

Proof

Proof of Lemma $\Rightarrow$ Corollary

What is the **complexity** of "$\forall f : \omega^\omega \to \omega^\omega$ continuous $\exists T \in \mathbb{IE}$ such that $f``[T] \in \mathcal{M}$"?

Proof

Proof of Lemma $\Rightarrow$ Corollary

What is the **complexity** of "$\forall f : \omega^\omega \to \omega^\omega$ continuous $\exists T \in \mathbb{IE}$ such that $f``[T] \in \mathcal{M}$"?

- "$f : \omega^\omega \to \omega^\omega$ is a continuous function" can be expressed as "$f' : \omega^{<\omega} \to \omega^{<\omega}$ is monotone and unbounded along each real", which is a $\mathbf{\Pi}^1_1$ statement with parameter f'.
- "$T \in \mathbb{IE}$" is arithmetic on the code of T.
- $f``[T]$ is an analytic set whose code is recursive in f' and T.
- For an analytic set to be meager is $\mathbf{\Pi}^1_1$.

So the statement "$\mathbb{IE}$ has the MIP" is $\mathbf{\Pi}^1_3$.

Now go to any forcing extension satisfying $\mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ (e.g add ω_2 Cohen reals), apply the lemma and conclude that $\mathbb{IE}$ had the MIP in the ground model by **downward $\mathbf{\Pi}^1_3$-absoluteness**. $\square$

Proofs

Proof of Lemma

- Let $\mathrm{add}(\mathfrak{I}_{\mathrm{ioe}}, \mathbb{IE})$ be the least size of a family $\{X_\alpha \mid \alpha < \kappa\}$ such that $X_\alpha \in \mathfrak{I}_{\mathrm{ioe}}$ but there is no $\mathbb{IE}$-tree T completely contained in the **complement** of $\bigcup_{\alpha<\kappa} X_\alpha$.
- Prove that $\mathrm{cov}(\mathcal{M}) \leq \mathrm{add}(\mathfrak{I}_{\mathrm{ioe}}, \mathbb{IE})$.
- Assume $\mathbb{IE}$ does **not** have the MIP: then there is $f : \omega^\omega \to \omega^\omega$ such that $f``[T]$ is not meager for all $T \in \mathbb{IE}$. This is equivalent to saying that f**-preimages of meager sets are** $\mathfrak{I}_{\mathrm{ioe}}$**-small**. From this it (essentially) follows that $\mathrm{add}(\mathfrak{I}_{\mathrm{ioe}}, \mathbb{IE}) \leq \mathrm{add}(\mathcal{M})$.
- This contradicts $\mathrm{add}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$. □

Homogeneity

Theorem (Kh-Laguzzi)

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But what we need is the MIP **below every** $S \in \mathbb{IE}$.

It would be sufficient for $\mathfrak{I}_{\mathrm{ioe}}$ to be **homogeneous** (the forcing as a whole is isomorphic to the part below a fixed condition).

Goldstern-Shelah tree

Recall the **full-splitting Miller** partial order $\mathbb{FM}$ from the wrong dichotomy. It is easy to see that $\mathbb{FM}$ adds Cohen reals.

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Lemma (Goldstern-Shelah 1994)

There exists $T^{GS} \in \mathbb{IE}$ such that **every** *$T \leq T^{GS}$ is an* **almost-full-splitting Miller tree**, *i.e., every t in T^{GS} has an extension s such that $\forall n \neq 0\ (s^\frown\langle n\rangle \in T)$.*

Construct T^{GS} in such a way that:

1. All splitting nodes of T^{GS} have different length, i.e., if $s, t \in \mathrm{Split}(T^{GS})$ and $s \neq t$ then $|s| \neq |t|$.
2. All $t \in T^{GS}$ which are **not** splitting satisfy $t(|t|-1) = 0$.

If $S \subseteq T^{GS}$ is an ioe-tree, this can **only** happen if every node can be extended to an almost-full-splitting one!

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1. $\mathfrak{I}_{\mathrm{ioe}}$ is very much **not** homogeneous.
2. "$\mathbb{IE}$ has the MIP below every condition" is **false.**
3. $T^{GS} \Vdash_{\mathbb{IE}}$ "there is a Cohen real".

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Consequences:

1. $\mathfrak{I}_{\mathrm{ioe}}$ is very much **not** homogeneous.
2. "$\mathbb{IE}$ has the MIP below every condition" is **false.**
3. $T^{GS} \Vdash_{\mathbb{IE}}$ "there is a Cohen real".

But could it be that $\exists T_0 \in \mathbb{IE}\, \forall S \leq T_0$ ($\mathbb{IE}$ has the MIP below S)?
Then T_0 would force that there are no Cohen reals.

On the other hand, if trees like T^{GS} are dense in $\mathbb{IE}$, then $\Vdash_{\mathbb{IE}}$ "there is a Cohen real".

Game

This is still an **open question**. We can formulate it in terms of a game:

I:		$S \leq T_0, f : [S] \to \omega^\omega$ continuous		...
II:	$T_0 \in \mathbb{IE}$		$T \leq S$	

	$(s_0, x(0))$		$(s_1, x(1))$		...
...		t_0		t_1	...

where $s_i, t_i \in \omega^{<\omega} \setminus \{\varnothing\}$ and $x(i) \in \omega$ are such that $x \in [T]$. Assuming all the rules are followed, Player I wins iff $f(x) = s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \dots$.

Lemma

If I wins then $\Vdash_{\mathbb{IE}}$ "there is a Cohen real". If II wins with first move T_0, then $T_0 \Vdash_{\mathbb{IE}}$ "there are no Cohen reals".

Question

Is there $T_0 \in \mathbb{E}$ forcing that no Cohen reals are added?

Kiitos huomiostanne!

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