Some Applications of Set Theory in Proof Theory

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- Essentially, ε-calculus $=$ propositional logic $+$ ε.

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- In the early 1900s, D. Hilbert investigated logic enhanced with built-in choice functions as part of his foundational program.
- This resulted in the ε-calculus.
- Essentially, ε-calculus = propositional logic + ε.
- More precisely, one adds to zeroth-order logic (that is, first-order logic without quantifiers) terms of the form $\varepsilon_x A(x)$, where 'x' is a (bound) variable.

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"from $A(t)$ for some t, infer $A(\varepsilon_x A(x))$."

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    - $A(\varepsilon_x A(x, \varepsilon_y A(x,y)), \varepsilon_y A(\varepsilon_x A(x, \varepsilon_y A(x,y)), y))$.

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- The ε-calculus: add to a Hilbert-style axiomatization of propositional logic all formulae of the form
  - $A(t) \to A(\varepsilon_x A(x))$, and
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- $\big(A(x) \leftrightarrow B(x)\big) \to \varepsilon_x A(x) = \varepsilon_x B(x)$ need not be an axiom.

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Can there be an infinitary analog of the ε-calculus? For example, can one find an analog of $\mathcal{L}_{\omega_1\omega_1}$?

- If so, it would need to have as axioms the translations of
  - $A(\vec{t}) \to \exists\vec{x}\, A(\vec{x})$, and
  - $\forall\vec{x}\, A(\vec{x}) \to A(\vec{t})$,

  where $\vec{t}$ (resp. $\vec{x}$) is a countable sequence of terms (resp. variables free in $A(\vec{x})$).

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- Recall that $\exists x \, \exists y \, A(x, y)$ was translated as $A(t_0, t_1)$, where
  - $t_0 = \varepsilon_x A(x, \varepsilon_y A(x, y))$,
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- There is a general pattern. For example, $\exists x\, \exists y\, \exists z\, A(x, y, z)$ is translated as $A(t_0, t_1, t_2)$, where letting
  - $s_0(y, z) = \varepsilon_x A(x, y, z)$,
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  we have
  - $t_0 = s_0(s_1(x, s_2(x, y)), s_2(x, y))$,
  - $t_1 = s_1(t_0, s_2(t_0, y))$,
  - $t_2 = s_2(t_0, t_1)$.

Infinitely deep terms

- This leads us to define the translation of $\exists x_0 \exists x_1 \ldots A(x_0, x_1, \ldots)$ as $A(t_0, t_1, \ldots)$, where
  - $s_i(x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots) = \varepsilon_{x_i} A(x_0, x_1, \ldots)$
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- The (Hilbert-style) infinite ε-calculus can be defined by adding to $\mathcal{L}_{\omega_1,0}$ the translations of all axioms of the form:
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  - $A(\vec{t}) \to \exists \vec{x}\, A(\vec{x})$, and
  - $\forall \vec{x}\, A(\vec{x}) \to A(\vec{t})$.
- (Convention: we assume that every atomic formula is of finite arity.)

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Theorem

Assume there are uncountably many Woodin cardinals. Then the infinite ε-calculus is conservative over (infinitary) propositional logic.

- It is to be expected that large cardinals are needed.
- This is because the language can express the determinacy of games of (fixed) countable length.

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- As before, one then derives $A(s, \varepsilon_y A(s,y))$ and, from it, the formula

$$A(\varepsilon_x A(x, \varepsilon_y (x,y)), \varepsilon_y A(\varepsilon_x A(x, \varepsilon_y (x,y), y)) \tag{1}$$

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- However, suppose that $A(x,y)$ is of the form $B(x, \varepsilon_z \neg B(x, z, \varepsilon_y B(x,y,z)), y)$.
- Then, (1) expresses something of the form $\exists x \, \forall z \, \exists y \, B(x,y,z)$.
- Thus, by only using rules that correspond to existential quantifiers, one can infer statements expressing infinite alternating strings of quantifiers.

Sequent Calculi

- A *sequent* is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are sequences of formulae.

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Sequent Calculi

- A *sequent* is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are sequences of formulae.
- It is to be interpreted as "if all the formulae in Γ are true, then some formula in Δ is true."
- One builds up proofs of sequents by using *rules*. For example:

$$\frac{\Gamma \vdash \Delta A}{\Gamma \vdash \Delta, A \vee B}$$

The Cut Rule

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$$\frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$$

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- Gentzen's consistency proof for Peano Arithmetic: he defined a sequent calculus that is sound and complete for arithmetic, LK. Then he proved the *cut-elimination theorem*:

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Theorem (Gentzen)

If a sequent is provable in LK, then it is provable without the cut-rule.

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Theorem

Let E be the reformulation of the infinite ε-calculus in terms of sequents. Then the following are equivalent:

1. *The ε-theorem holds for E.*
2. *The cut-elimination theorem holds for E.*
3. *All games of countable length with projective payoff are determined.*

Cut Elimination

One possible proof is based on interpreting a suitable first-order proof system inside E.

Theorem

There is an infinitary first-order sequent calculus F such that the following are equivalent:

1. *The cut-elimination theorem holds for F.*
2. *All games of countable length with projective payoff are determined.*

Cut Elimination

This in turn is based on a similar construction by Takeuti.

Theorem (Takeuti, 1970s)

There is an infinitary first-order sequent calculus D such that the following are equivalent for any transitive model M of ZF+DC:

1. $M \models$ *"The cut-elimination theorem holds for D."*
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Takeuti's method also yields analogous results for, say, $AD_{\mathbb{R}}$ or PD.

Thank you.