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On definability of team relations with k-invariant atoms

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Presentation in a nutshell

1. We introduce a semantic notion of k-invariance for atoms with team semantics.
2. We present a theorem for proving various undefinability results for logics extending FO with k-invariant atoms.
3. We sketch a proof for this theorem and introduce various useful definitions along the way.

Setting

- We consider *first-order logic with team semantics* extended with arbitrary atomic formulae.
- We assume *lax semantics* for disjuction and existential quantifier (in order to maintain locality).

Team relations

Let $\mathcal{M}$ be a model and let X be a *team* for $\mathcal{M}$.

For any $\{v_1, \ldots, v_k\} \subseteq \mathrm{dom}(X)$ we write

$$X(v_1 \ldots v_k) = \{s(v_1 \ldots v_k) \mid s \in X\}.$$

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$$X(v_1 \ldots v_k) = \{s(v_1 \ldots v_k) \mid s \in X\}.$$

Hence every k-tuple $\vec{v}$ of variables in $\mathrm{dom}(X)$ naturally defines a corresponding k-ary *team relation* $X(\vec{v})$ in $\mathcal{M}$.

Expressive power and definability of team relations

By saying that a class $\mathcal{P}$ (i.e. a property) of k-ary team relations is *definable* in a logic L with team semantics, we mean that by fixing a tuple $v_1 \ldots v_k$ of distinct variables (in the given order) there is a formula $\varphi(v_1 \ldots v_k) \in \mathrm{L}$ s.t.

$$\mathcal{M} \models_X \varphi \text{ iff } X(v_1 \ldots v_k) \in \mathcal{P}.$$

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Note that for example the property of *symmetry* is FO-definable as a property of a *relation* $R^{\mathcal{M}}$ *in a model*, but it is undefinable as a property of a *team relation* $X(v_1 v_2)$ in downwards closed logics.

k-equivalence

Let X, Y be teams which have a shared domain D.

We say that X and Y are *k-equivalent* if they have the *same k-ary team relations*, i.e. the following holds for all $\{v_1, \ldots, v_k\} \subseteq D$:

$$X(v_1 \ldots v_k) = Y(v_1 \ldots v_k).$$

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Example

The following teams

$$X := \{\{(x,0),(y,0)\},\{(x,1),(y,1)\}\}$$
$$Y := \{\{(x,1),(y,0)\},\{(x,0),(y,1)\}\}$$

are 1-equivalent since $X(x) = Y(x)$ and $X(y) = Y(y)$,
but they are not 2-equivalent since $X(xy) \neq Y(xy)$.

k-invariance

We say that an atom A is k*-invariant* if we have

$$\mathcal{M} \models_X A \quad \text{iff} \quad \mathcal{M} \models_Y A,$$

for all models $\mathcal{M}$ and k-equivalent teams X and Y for $\mathcal{M}$.

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Example: Unary inclusion atom $v_1 \subseteq v_2$ is 1-invariant:

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Note that the notion of k-invariance is very "liberal" since it allows e.g. atoms that are not invariant under isomorphisms.

Class $\mathcal{L}[k]$ of logics with team semantics

For $k \geq 1$, a logic L belongs to class $\mathcal{L}[k]$ if

(1) L is an extension of FO with atomic formulas s.t. L is local;

(2) all atomic formulas in L belong to either (or both) of the following two classes:

  (a) downwards closed atoms;
  (b) k-invariant atoms.

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In particular, all k-ary fragments of logics with team semantics (studied so far) belong to the class $\mathcal{L}[k]$.

On the expressive power of k-ary inclusion-exclusion logic

k-ary inclusion-exclsusion logic (INEX[k]) is obtained by adding k-ary inclusion and exclusion atoms to FO. We know that

1. All INEX[k]-definable properties of team relations are ESO[k]-definable.
2. All ESO[k]-definable properties of at most k-ary team relations are INEX[k]-definable.

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The case $k = 1$ with binary team relations is particularly interesting; for example, can the following properties be defined in INEX[1]?

(a) $X(y_1 y_2)$ is symmetric.

(b) $X(y_1 y_2)$ is c-colorable for a given $c \geq 1$.

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Surprisingly, (b) is definable INEX[1], but (a) is not.

Undefinability theorem for $\mathcal{L}[k]$

Theorem

Let $\mathcal{P}$ be any property of $(k+1)$-ary relations.

Assume that there is a constant c such that for any finite model $\mathcal{M}$, with at least c elements, there are teams X and X^* for $\mathcal{M}$ s.t. the following conditions hold:

1. $X^* \subseteq X$
2. $\mathrm{dom}(X) = \{y_1, \ldots, y_{k+1}\}$.
3. $X(y_1 \ldots y_{k+1})$ *has the property* $\mathcal{P}$.
4. $(X \setminus \{s\})(y_1 \ldots y_{k+1})$ *does not have* $\mathcal{P}$ *for any* $s \in X^*$.
5. $|X^*| \geq \frac{|\mathrm{dom}(\mathcal{M})|^{k+1}}{c}$.

Then the property $\mathcal{P}$ cannot be defined in any logic $\mathrm{L} \in \mathcal{L}[k]$.

Simple applications of the undefinability theorem (1/3)

$(k+1)$-*totality* of $X(y_1 \ldots y_{k+1})$ is undefinable in $\mathcal{L}[k]$.

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Proof.

Let $c = 1$ and let $\mathcal{M}$ be any finite model. Let X be the team s.t. $\mathrm{dom}(X) = \{y_1, \ldots, y_{k+1}\}$ and $X(y_1 \ldots y_{k+1}) = \mathrm{dom}(\mathcal{M})^{k+1}$ and let $X^* = X$. The claim follows now immediately from the undefinability theorem. □

Simple applications of the undefinability theorem (2/3)

Symmetry of $X(y_1 y_2)$ cannot be defined in $\mathcal{L}[1]$.

Proof.

Let $c = 2$ and let $\mathcal{M}$ be any finite model with at least 2 elements.

Let X be the team for $\mathcal{M}$ s.t. $\mathrm{dom}(X) = \{y_1, y_2\}$ and $X(y_1 y_2) = \mathrm{dom}(\mathcal{M})^2$. Let

$$X^* := \{s \in X \mid s(y_1) \neq s(y_2)\}.$$

Now $X(y_1 y_2)$ is symmetric, but $(X \setminus \{s\})(y_1 y_2)$ is not symmetric for any $s \in X^*$. We also have

$$|X^*| = |\mathrm{dom}(\mathcal{M})|^2 - |\mathrm{dom}(\mathcal{M})| \geq \frac{|\mathrm{dom}(\mathcal{M})|^2}{2} = \frac{|\mathrm{dom}(\mathcal{M})|^2}{c}.$$

□

Simple applications of the undefinability theorem (3/3)

$X(y_1y_2)$ being a *linear order* cannot be defined in $\mathcal{L}[1]$.

Proof.

Let $c = 2$ and let $\mathcal{M}$ be any finite model with at least 2 elements.

Let X be any team for $\mathcal{M}$ such that $\mathrm{dom}(X) = \{y_1, y_2\}$ and $X(y_1y_2)$ is a linear order (on $\mathcal{M}$).

Let $X^* = X$; now $(X \setminus \{s\})(y_1y_2)$ is not a linear order for any $s \in X^*$. We also have

$$|X^*| = |X| = \frac{|M|^2 - |M|}{2} + |M| > \frac{|M|^2}{2} = \frac{|M|^2}{c},$$

where $M := \mathrm{dom}(\mathcal{M})$. □

Sketching a proof
for the undefinability theorem

Evaluations

Hereafter let L be any logic in $\mathcal{L}[k]$ and let φ be a formula in L.

Let $\mathcal{E}(\mathcal{M}, X, \varphi)$ be the class of functions, called *evaluations*, which are mappings E from the subformulas of φ to teams for $\mathcal{M}$ s.t.

- $E(\varphi) = X$.
- $E(\psi \wedge \theta) = E(\psi) = E(\theta)$.
- $E(\psi \vee \theta) = E(\psi) \cup E(\theta)$.
- $E(\exists x\, \psi)[F/x] = E(\psi)$ for some function $F : E(\psi) \to \mathcal{P}(\mathrm{dom}(\mathcal{M})) \setminus \{\emptyset\}$.
- $E(\forall x\, \psi)[A/x] = E(\psi)$.

(Similar approach has been used by [Ebbing, Hella, Lohmann, Virtema 2017] in the context of *Boolean dependence logic*.)

Satisfying evaluations

We call $E \in \mathcal{E}(\mathcal{M}, X, \varphi)$ *a satisfying evaluation* if we have

$$\mathcal{M} \vDash_{E(\psi)} \psi \text{ for each } \psi \in \mathrm{dom}(E).$$

The set of all satisfying evaluations $E \in \mathcal{E}(\mathcal{M}, X, \varphi)$ is denoted by $\mathrm{Sat}(\mathcal{M}, X, \varphi)$.

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It is clear that $\mathcal{M} \vDash_X \varphi$ if and only if $\mathrm{Sat}(\mathcal{M}, X, \varphi) \neq \emptyset$.

Removal of single assignments and their extension sets

Let s be an assignment and let Y be a team s.t. $\mathrm{dom}(s) \subseteq \mathrm{dom}(Y)$. The *extension set of* s *in* Y, denoted by $Y_{s\prec}$, is defined as follows:

$$Y_{s\prec} := \{r \in Y \mid r \upharpoonright \mathrm{dom}(s) = s\}.$$

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Lemma

Let $s \in X$ and $E \in \mathrm{Sat}(\mathcal{M}, X, \varphi)$. Suppose that

$$E(\psi) \setminus E(\psi)_{s\prec} \text{ is } k\text{-equivalent to } E(\psi),$$

for each k-invariant atom ψ in φ.

Then it holds that $\mathcal{M} \vDash_{X\setminus\{s\}} \varphi$.

Estimate for the number of "k-separating assignments"

Let $E \in \mathcal{E}(\mathcal{M}, X, \varphi)$.

Let $\mathrm{Sep}_X^k(E)$ denote the set of those assignments s in X s.t.

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It can be shown that

$$|\mathsf{Sep}^k_X(E)| \;\leq\; (|\mathrm{dom}(X)| + |\varphi|)^k \cdot |\mathrm{dom}(\mathcal{M})|^k.$$

Removal lemma

Lemma

Let $m, k, c \geq 1$ *and let* $\mathcal{M}$ *be a model s.t.*

$$\mathrm{dom}(\mathcal{M}) = \{1, \ldots, c \cdot p^{3k}\},$$

where $p = \max(m+1, k, 3)$.

Let X,X^* *be teams for* $\mathcal{M}$ *s.t.* $X^* \subseteq X$, $|\mathrm{dom}(X)| = k+1$ *and*

$$|X^*| \geq \frac{|\mathrm{dom}(\mathcal{M})|^{k+1}}{c}.$$

Then for every $\varphi \in \mathrm{L}$, *s.t.* $|\varphi| \leq m$, *the following implication holds:*

$$\textit{If } \mathcal{M} \models_X \varphi, \textit{ then there exists } s \in X^* \textit{ s.t. } \mathcal{M} \models_{X \setminus \{s\}} \varphi.$$

Undefinability theorem for $\mathcal{L}[k]$ (revisited)

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Then the property $\mathcal{P}$ cannot be defined in any logic $\mathrm{L} \in \mathcal{L}[k]$.

Thank you for your attention!