Algebraic Semantics of Propositional Dependence Logic

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Motivations

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Can we give an alternative, algebraic semantics, to propositional dependence logic? This would allow us to apply categorical and algebraic methods in the study of dependence logic.

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- Quadrellaro 2019 and Grilletti and Quadrellaro 2020 generalize this semantics to so-called DNA-logics and χ-variants of intermediate logics.

Preliminaries

Characterisation of Downward Team Algebras

Equivalence of Team and Algebraic Semantics

Syntax

Let AT be a countable set of atomic propositions, then the language of propositional dependence logic $\mathcal{L}_D$ is defined as follows:

$$\phi ::= p \mid \bot \mid =\!(p) \mid \psi \wedge \chi \mid \psi \otimes \chi \mid \psi \to \chi$$

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Two Key Properties

Theorem (Downward Closure)

Let s, t be two teams such that $s \subseteq t$ and $\varphi \in \mathcal{L}_D$, then if $t \vDash \varphi$ then $s \vDash \varphi$.

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Theorem (Finite Model Property)

Suppose $\wp(2^{\mathtt{AT}}) \nvDash \varphi$, then there is a finite team $t \in \wp(2^{\mathtt{AT}})$ such that $t \nvDash \varphi$.

Lattices

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Heyting Algebras

We say that a bounded distributive lattice H is a Heyting algebra if for every $x, y \in H$ there is some element $a \to b \in H$ such that for all $z \in H$:

$$z \leq x \to y \Leftrightarrow z \wedge x \leq y$$

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An ND-algebra is a Heyting algebra H which also satisfies the following equation, for every $k \geq 2$:

$$\neg p \to \bigvee_{i \leq k} \neg q_i = \bigvee_{i \leq k} (\neg p \to \neg q_i)$$

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- H is regular if $H = \langle H_\neg \rangle$.

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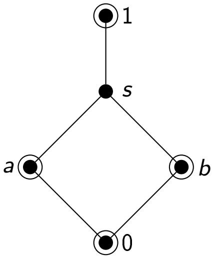
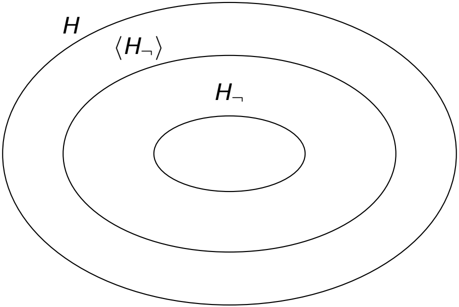
Our Favourite Example of a Heyting Algebra



Regular and Non-Regular Heyting Algebras



Definition (Downward Team Algebra)

A *downward team algebra* H is a tuple $(H, \vee, \otimes, \wedge, \to, 0, H_\neg, \vee_B)$, where $(H, \vee, \wedge, \to, 0)$ is a $\mathtt{ND}$-algebra, $(H_\neg, \vee_B, \wedge, \to, 0)$ a Boolean algebra and, in addition, it satisfies the following equations:

$$(*) \quad x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$$
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An *algebraic downward team model* of **PD** is a pair $M = (H, V^{\neg})$ such that H is a **DTA** and $V^{\neg}$ a team valuation.

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Remark: the reason we consider *team valuations* is to mirror the semantics of atomic propositions in team semantics.

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- (*Validity*) $\mathbf{DTA} \vDash^{\neg} \phi$ if for all $H \in \mathbf{DTA}$, $H \vDash^{\neg} \phi$.

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We then have two important theorems by Birkhoff:

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Notice: all the algebraic structures we have introduced before are definable by equations, hence these two theorems apply in our context. We can also prove a stronger version of the second Birkhoff Theorem:

Theorem

Let $\phi \in \mathcal{L}_D$, then $\mathbf{DTA} \not\models^{\neg} \phi$ *entails* $\mathbf{DTA}_{FRSI} \not\models^{\neg} \phi$.

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Characterisation of Downward Team Algebras

Equivalence of Team and Algebraic Semantics

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Our goal is to prove the following theorem:

Theorem

The categories **FinSet**, $Var_{FRSI}(\mathtt{ND})$ *and* $\mathbf{DTA}_{FRSI}$ *are equivalent:*

$$\mathbf{FinSet} \cong Var_{FRSI}(\mathtt{ND}) \cong \mathbf{DTA}_{FRSI}.$$

From **FinSet** to $Var_{FRSI}(\mathtt{ND})$, and Back

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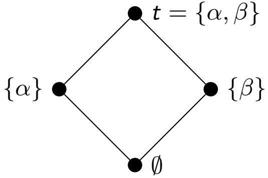

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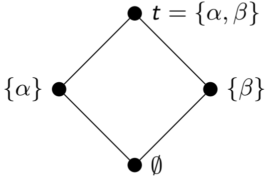

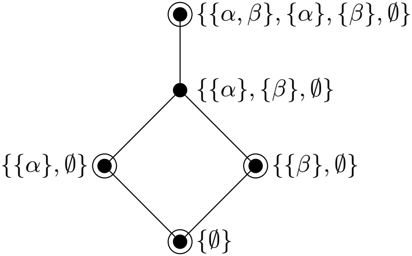

From **FinSet** to $Var_{FRSI}(\mathtt{ND})$, and Back

Theorem (Bezhanishvili, Grilletti, Holliday)

Let H be an Heyting algebra. Then $H \in Var_{FRSI}(\mathtt{ND})$ iff there is some finite set s such that $H \cong H_s$.

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Corollary

The categories **FinSet** *and $Var_{FRSI}(\mathtt{ND})$ are equivalent.*

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Sketch: Let $F : \mathbf{DTA}_{FRSI} \to Var_{FRSI}(\mathtt{ND})$ be the forgetful functor sending every **DTA** to its underlying ND-algebra.

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Define the functor $G : Var_{FRSI}(\mathtt{ND}) \to \mathbf{DTA}_{FRSI}$ sending every f.r.s.i. ND-algebra to the **DTA** obtained by adding an operator $\otimes$ defined in the following way:

$$x \otimes y := \bigvee \{a \vee_B b : a \leq x, b \leq y \text{ and } a, b \in H_\neg\}.$$

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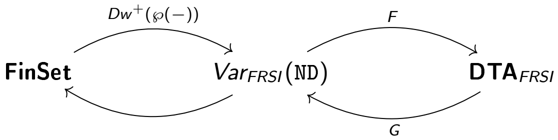

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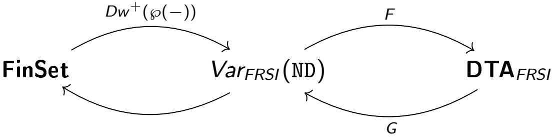

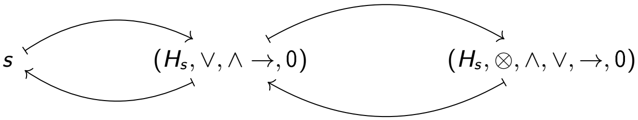

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Proposition

Let $s \in \wp(2^{\mathtt{AT}})$ be a finite team and $M_s = (H_s, V_s^{\neg})$ its corresponding f.r.s.i ***DTM****. We have that $s \vDash \phi$ if and only if $M_s \vDash^{\neg} \phi$.*

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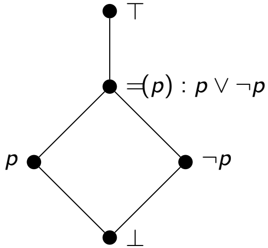

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Proposition

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[*Proof Idea: by previous direction and categorical equivalence*]

Main Result

Theorem (Equivalence of Team and Algebraic Semantics)

The team semantics and the algebraic semantics of **PD** *are equivalent, i.e.* $\wp(2^{\mathtt{AT}}) \vDash \phi$ *if and only if* **DTA** $\vDash^{\neg} \phi$.

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($\Leftarrow$) Suppose $\wp(2^{\mathtt{AT}}) \not\models \phi$, then by the finite model property there is some finite team s such that $s \not\models \phi$. By the previous Proposition it follows that $M_s \not\models^{\neg} \phi$.

□

Continuation: Algebraic Semantics for Team-Based Logics

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An *intuitionistic downward team algebra* H is a tuple $(H, \vee\!\!\!\vee, \vee, \wedge, \to, 0, H_0)$, where:

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- For all $x, y, z \in H$, the following equality hold:

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Remark: here the core is not definable!

Thank you for your attention!

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