Dependency: the question-based view

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In the dependence logic tradition, dependency is construed as a relation between variables (V-dependency).

I want to discuss a different conception, grounded in work on inquisitive logic: dependency as a relation between questions (Q-dependency).

V-dependency

Let T be a team—a set of assignments.

$$T = \begin{array}{|c|cccccc|} \hline \mathrm{x} & 1 & 1 & 2 & 2 & 3 & 3 \\ \mathrm{y} & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline \end{array}$$

Meta-language

$$\mathbb{D}_T(x, y) \iff \forall g, g' \in T : g(x) = g'(x) \Rightarrow g(y) = g'(y)$$

Object language formula

$$T \models \mathsf{D}(x, y) \iff \mathbb{D}_T(x, y)$$

x	1	1	2	2	3	3
y	0	0	1	1	1	1

Equivalent formulation

$\mathbb{D}_T(x, y) \iff \forall T' \subseteq T : x$ constant in $T' \Rightarrow y$ constant in T'

x	1	1	2	2	3	3
y	0	0	1	1	1	1

Equivalent formulation

$$\begin{aligned}\mathbb{D}_T(x,y) &\iff \forall T' \subseteq T : x \text{ constant in } T' \Rightarrow y \text{ constant in } T' \\ &\iff \forall T' \subseteq T : T' \text{ settles what is the value of } x \Rightarrow \\ &\qquad\qquad\qquad\qquad T' \text{ settles what is the value of } y\end{aligned}$$

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Equivalent formulation

$$\mathbb{D}_T(x,y) \iff \forall T' \subseteq T : x \text{ constant in } T' \Rightarrow y \text{ constant in } T'$$
$$\iff \forall T' \subseteq T : T' \text{ settles what is the value of } x \Rightarrow T' \text{ settles what is the value of } y$$

Support for questions

$T \models Q \iff Q$ is settled in T

Example: value questions

- Vx : what is the value of x?
- $T \models Vx \iff \forall g, g' \in T : g(x) = g'(x)$

x	1	1	1	1
y	2	2	3	3

$T \models Vx \qquad T \not\models Vy$

Q-dependency

Meta-language relation

$$Q \rightsquigarrow_T Q' \iff \forall T' \subseteq T : T' \models Q \Rightarrow T' \models Q'$$

Object language formula

$$T \models Q \to Q' \iff Q \rightsquigarrow_T Q'$$

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Dependency of value questions

$$\begin{aligned} Vx \rightsquigarrow_T Vy &\iff \forall T' \subseteq T : T' \models Vx \Rightarrow T' \models Vy \\ &\iff \forall T' \subseteq T : x \text{ constant in } T' \Rightarrow y \text{ constant in } T' \\ &\iff \mathbb{D}_T(x, y) \end{aligned}$$

We capture the same property of a team, but as a relation between questions, rather than as a relation between variables.

What are the reasons to be interested in this alternative perspective?

I'll discuss three:

1. flexibility
2. generality
3. logicality

1. Flexibility

Questions are sentences; as such, they can be combined by logical operators.

Two examples where this is convenient:

- Conjunction
- Conditionalization

Conjunction

Both dependency relations can be generalized naturally to the case of multiple determinants.

$$\mathbb{D}_T(x_1, \ldots, x_n; y) \iff \forall g, g' \in T : \text{if } g(x_i) = g'(x_i) \text{ for all } i \leq n \text{ then } g(y) = g'(y)$$

$$Q_1, \ldots, Q_n \rightsquigarrow_T Q' \iff \forall T' \subseteq T : \text{ if } T' \models Q_i \text{ for all } i \leq n \text{ then } T' \models Q'$$

In the case of Q-dependency, however, the general version is reducible to the binary version relying on conjunction.

Question conjunction

$T \models Q \wedge Q' \iff T \models Q$ and $T \models Q'$

Reduction

$Q_1, \ldots, Q_n \rightsquigarrow_T Q' \iff Q_1 \wedge \cdots \wedge Q_n \rightsquigarrow_T Q'$

Object language

In the object language we can make do with a binary connective $\rightarrow$:

$T \models Q_1 \wedge \cdots \wedge Q_n \rightarrow Q' \iff Q_1, \ldots, Q_n \rightsquigarrow_T Q'$

Conditionalization

Below, y is the Fibonacci number of x. Then y determines x provided $y \geq 2$.
What we have is a conditional dependency.

x	1	2	3	4	5	6	...
y	1	1	2	3	5	8	...

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To capture this we could generalize the relation $\mathbb{D}$ even further:

$$\mathbb{D}_T(S;\, x_1, \ldots, x_n;\, y) \iff \forall g, g' \in T \cap |S| \quad \text{where } |S| = \{g \mid M \models_g S\}$$
$$\text{if } g(x_i) = g'(x_i) \text{ for all } i \leq n$$
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We then need a corresponding generalized operator $D(S; x_1, \ldots, x_n; y)$.

From the Q-dependency perspective, things are simpler.

Support for statements

$T \models S \iff T \subseteq |S| \qquad$ where $|S| = \{g \mid M \models_g S\}$

Then we need no special "slot" for statements in the dependency relation $\rightsquigarrow$. Statements and questions can be treated on a par as determinants:

$$
\begin{aligned}
S, Q \rightsquigarrow_T Q' \quad &\iff \quad \forall T' \subseteq T : T' \models S \text{ and } T' \models Q \Rightarrow T' \models Q' \\
&\iff \quad \forall T' \subseteq T : T' \subseteq |S| \text{ and } T' \models Q \Rightarrow T' \models Q' \\
&\iff \quad \forall T' \subseteq T \cap |S| : T' \models Q \Rightarrow T' \models Q' \\
&\iff \quad Q \rightsquigarrow_{T \cap |S|} Q'
\end{aligned}
$$

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We get conditional dependencies for free without generalizing the relation $\rightsquigarrow_T$, simply by plugging in statements as determinants.

Example

Our observation above amounts to the relation:

$$y \geq 2,\ Vy \rightsquigarrow_T Vx$$

x	1	2	3	4	5	6	...
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or equivalently as

$$(y \geq 2) \rightarrow (Vy \rightarrow Vx)$$

Main point

The Q-dependency relation $\rightsquigarrow_T$ connects sentences.

Conjunction

- Sentences can be conjoined.
- We can use $\wedge$ to combine multiple determinants:

$$Q \wedge Q' \rightarrow Q''$$

Conditionalization

- Statements are also sentences.
- Having them as determinants gives us conditional dependencies for free.

$$S \wedge Q \rightarrow Q'$$

2. Generality

Example

x	3	3	4	4	5	5	6	6
y	2	4	3	5	4	6	5	7

No non-trivial V-dependencies hold:

- We have neither $\mathbb{D}_T(x, y)$ nor $\mathbb{D}_T(y, x)$.

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- Value of x together with whether $x < y$ determines the value of y.

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- ...

These can be captured as instances of the Q-dependence relation once we look beyond the special case of value questions.

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Polar questions

If S is a statement, we can form the polar question $?S$ ('whether S').

$$T \models ?S \iff S \text{ has the same truth-value at all } g \in T$$

Example

$$T \models ?\mathrm{Even}(x) \iff \forall g \in T : g(x) \text{ is even or } \forall g \in T : g(x) \text{ is odd}$$

The relations observed above can be captured as Q-dependencies:

x	3	3	4	4	5	5	6	6
y	2	4	3	5	4	6	5	7

Whether x is prime determines whether it is even

- Meta-language: $?\mathrm{Prime}(x) \rightsquigarrow_T ?\mathrm{Even}(x)$
- Object language: $?\mathrm{Prime}(x) \rightarrow ?\mathrm{Even}(x)$

Value of x together with whether $x < y$ determines value of y:

- Meta-language: $Vx,\ ?(x < y) \rightsquigarrow_T Vy$
- Object language: $Vx \wedge ?(x < y) \rightarrow Vy$

Another example

Value of x determines the odd prime factors of y.

x	1	1	2	2	3	3	4	4
y	2	4	3	6	5	10	15	30

Which questions

- $OF(z, y) =$ "z is an odd prime factor of y"
- $Wz.OF(z, y) =$ " what are the odd prime factors of y?"
- $T \models Wz.OF(z, y) \iff \forall g, g' \in T : OF[g(y)] = OF[g'(y)]$

Value of x determines the odd prime factors of y

- Meta-language: $Vx \rightsquigarrow_T Wz.OF(z, y)$
- Object language: $Vx \rightarrow Wz.OF(y, z)$

Main point

- We have a dependency whenever information of a type (input type) yields information of another type (output type).
- Questions can be seen as names for information types:
  - Vx : value of x
  - $?Even(x)$: parity of x
  - $WyOF(x, y)$: odd prime factors of x
- To represent an arbitrary dependence, we just have to find questions Q and Q' that capture the relevant input and output types.
- Then the dependency amounts to $Q \rightsquigarrow_T Q'$.

3. Logicality

Recall

$$A_1, \ldots, A_n \rightsquigarrow_T C \iff \forall T' \subseteq T : (T' \models A_i \text{ for all } i) \Rightarrow T' \models C$$

3. Logicality

Recall

$A_1, \ldots, A_n \rightsquigarrow_T C \iff \forall T' \subseteq T : (T' \models A_i \text{ for all } i) \Rightarrow T' \models C$

Observation

$\rightsquigarrow_T$ is a consequence relation (satisfies reflexivity, transitivity, and weakening). Moreover, for all sentences A, B, C we have:

$$A, B \rightsquigarrow_T C \iff A \wedge B \rightsquigarrow_T C \iff A \rightsquigarrow_T B \to C$$

Logical consequence

$\models \; := \bigcap_{M,T} (\rightsquigarrow_T)$

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Conservativity

In restriction to statements, $\models$ coincides with standard logical entailment:

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$S \models S' \iff \forall M, g : (M \models_g S \Longrightarrow M \models_g S')$

Logical dependency

$S, Q \models Q'$ expresses the fact that given S, Q logically determines Q'.

Examples

- $Px \leftrightarrow \neg Qy$, $?Px \models ?Qy$
- $y = f(x)$, $Vx \models Vy$

So, we can generalize the classical notion of entailment to questions: logical dependency turns out to be a facet of entailment.

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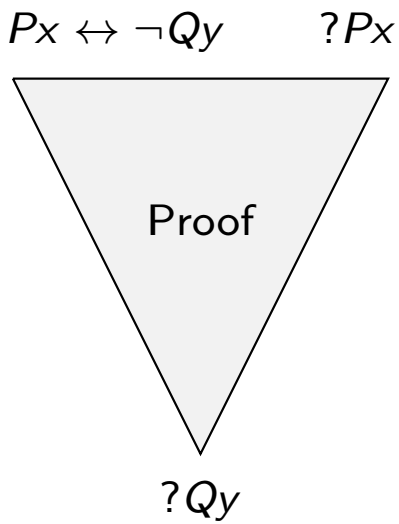
This also has many repercussions.

Example

Valid entailments are the sort of thing we prove in logic. If we have a proof system for this generalized entailment, we can prove logical dependencies.



Implication

The connective $\rightarrow$ behaves like an implication for this notion of entailment:
$A, B \models C \iff A \models B \rightarrow C$

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This means that we can make inferences with dependence formulas by the standard rules for implication:

$$\frac{\begin{array}{c}[Q] \\ \vdots \\ Q'\end{array}}{Q \to Q'}\ (\to i) \qquad\qquad \frac{Q \quad Q \to Q'}{Q'}\ (\to e)$$

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So the connection between dependence formulas and implications is not just notational, but substantial.

Conservativity

When S, S' are statements, $S \to S'$ coincides with the material conditional.

Main point

By viewing dependency as a relation between sentences, we can see that it is deeply connected to the central concerns of logic:

- logical dependency is a generalization to questions of logical entailment;
- the dependency operator $\rightarrow$ is a generalization of classical implication.

From approach to systems

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- My presentation has been vague: I have not defined a specific system.
- This was on purpose: what I wanted to present is a general approach to dependence, a template that can have many concrete realizations.
- The approach itself determines certain features of a system (team semantics, treatment of $\wedge, \rightarrow$) but leaves much to be specified.

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- This was on purpose: what I wanted to present is a general approach to dependence, a template that can have many concrete realizations.
- The approach itself determines certain features of a system (team semantics, treatment of $\wedge, \rightarrow$) but leaves much to be specified.
- In fact, I should have been more vague: I have taken teams to be sets of assignments, but the approach works for any set of classical points of evaluation (propositional valuations, possible worlds, etc. ...).

Next, I want to look at a specific system: inquisitive first-order logic (InqBQ).

This brief incursion has three aims:

- illustrate the ideas above in a concrete setting;
- mention two results from Gianluca Grilletti's forthcoming dissertation;
- to advertise some major open questions.

Language

$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x \varphi \mid \varphi \mathbin{\mathbb{\vee}} \varphi \mid \overline{\exists} x \varphi$ with p a FOL atom

Classical language

Formulas without $\mathbb{\vee}, \overline{\exists}$ are called classical and identified with FOL formulas, where $\neg, \vee, \exists$ are taken as defined operators:

- $\neg\varphi := \varphi \to \bot$
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
- $\exists x \varphi := \neg\forall x \neg\varphi$

Inquisitive operators

We read the inquisitive operators $\mathbb{\vee}, \overline{\exists}$ as question-forming operators. E.g.:

- $?\alpha := \alpha \mathbin{\mathbb{\vee}} \neg\alpha \approx$ "whether α or $\neg\alpha$"
- $\overline{\exists} x \alpha(x) \approx$ "what is an x such that $\alpha(x)$"

Models

$M = \langle W, D, I \rangle$ where I equips each $w \in W$ with a rel. structure I_w over D.

In InqBQ we are interested in questions and dep. about what the world is like. Hence, the relevant 'team' is not a set of assignments, but a set of worlds.

Semantics

Relation $s \models_g \varphi$ where $s \subseteq W$ and $g : \mathrm{Var} \to D$.

Semantic clauses

- $s \models_g p \iff \forall w \in s : I_w \models_g p$
- $s \models_g \bot \iff s = \emptyset$
- $s \models_g \varphi \wedge \psi \iff s \models_g \varphi$ and $s \models_g \psi$
- $s \models_g \varphi \mathbin{\overline{\vee}} \psi \iff s \models_g \varphi$ or $s \models_g \psi$
- $s \models_g \varphi \to \psi \iff \forall t \subseteq s : t \models_g \varphi \Longrightarrow t \models_g \psi$
- $s \models_g \forall x \varphi \iff \forall d \in D : s \models_{g[x \mapsto d]} \varphi$
- $s \models_g \overline{\exists} x \varphi \iff \exists d \in D : s \models_{g[x \mapsto d]} \varphi$

Classical formulas are 'flat'

For classical formulas α, support amounts to global truth:
$s \models_g \alpha \iff \forall w \in s : I_w \models_g \alpha$ in standard FOL

Conservativity

Restricted to classical formulas, entailment coincides with FOL-entailment.

Questions

Using $\bigvee, \overline{\exists}$ we can express a broad repertoire of interesting questions.

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Using $\mathbb{V}, \overline{\exists}$ we can express a broad repertoire of interesting questions.

Polar questions

- $?Pc \approx$ does c have property P?
- $s \models_g ?Pc \iff Pc$ has the same truth-value in all $w \in s$

Mention-all questions

- $\forall x?Px \approx$ which objects are P?
- $s \models_g \forall x?Px \iff P$ has the same extension in all $w \in s$.

Mention-some questions

- $\overline{\exists} xPx \approx$ what is an instance of P?
- $s \models_g \overline{\exists} xPx \iff \exists d \in D$ s.t. $\forall w \in s : d \in P_w$

Expressing dependencies

Given any two questions Q, Q' expressible in the system, we can express the corresponding dependence by $Q \to Q'$. (Cf. the generality point above)

Examples

- $\forall x?Px \to \forall x?Qx$
  The extension of P determines the extension of Q.
- $\overline{\exists} xPx \to \overline{\exists} xQx$
  Any instance of P determines an instance of Q.
- $\exists xPx \to (\forall x?Px \to \overline{\exists} xQx)$
  Conditionally on the existence of a P, the extension of P determines an instance of Q.

Logical dependencies

Dependencies that hold in all states come out as valid entailments.
Examples:

- $\forall x(Px \leftrightarrow \neg Qx), \forall x?Px \models \forall x?Qx$
- $\exists xPx, \forall x?Px \models \overline{\exists} xPx$
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Proof system

A (candidate) natural deduction system for InqBQ consists of:

- standard intuitionistic i/e rules for each operator
- double negation for classical formulas
- a few specific inquisitive principles

Example

$\forall x(Px \leftrightarrow \neg Qx),\ \forall x?Px \models \forall x?Qx$

$$\dfrac{\dfrac{\forall x?Px}{?Py}\ \forall\mathsf{e} \qquad \dfrac{\dfrac{\dfrac{\forall x(Px \leftrightarrow \neg Qx)}{Py \leftrightarrow \neg Qy}\ \forall\mathsf{e} \qquad \dfrac{\dfrac{\dfrac{\forall x(Px \leftrightarrow \neg Qx)}{Py \leftrightarrow \neg Qy}\ \forall\mathsf{e}}{\neg Qy \rightarrow Py}\ \wedge\mathsf{e} \qquad [\neg Qy]}{Py}\ \rightarrow\mathsf{e} \qquad [\neg Py]}{\bot}\ \rightarrow\mathsf{i}}{\neg\neg Qy}\ \mathsf{DNE}}{Qy}\ \vee\mathsf{i}}{?Qy}\ \vee\mathsf{e}$$

$$\dfrac{\dfrac{\dfrac{\forall x?Px}{?Py}\ \forall\mathsf{e} \qquad \dfrac{\dfrac{\forall x(Px \leftrightarrow \neg Qx)}{Py \leftrightarrow \neg Qy}\ \forall\mathsf{e} \qquad [Py]}{Py \rightarrow \neg Qy}\ \wedge\mathsf{e}}{\neg Qy}\ \rightarrow\mathsf{e}}{?Qy}\ \vee\mathsf{i}}{\forall x?Qx}\ \forall\mathsf{i}$$

Question:
Is the candidate proof system complete?

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Theorem (Grilletti, forthcoming)
The system is strongly complete for the classical antecedent fragment, i.e., the fragment where only classical formulas are allowed as antecedents of $\rightarrow$.

For instance, all logical dependencies above fall within the fragment, so we know they must be provable:

- $\forall x(Px \leftrightarrow \neg Qx), \forall x?Px \models \forall x?Qx$
- $\exists xPx, \forall x?Px \models \overline{\exists} xPx$
- $\forall x?Px \models ?\exists xPx$

Question (Axiomatizability)

Abstracting away from the particular proof system, we can ask:
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Both are currently OPEN

Connection with powerset

The difficulties are brought about by $\rightarrow$, which quantifies over elements of $\wp(s) = \{t \mid t \subseteq s\}$.

$$\begin{aligned} s \models \varphi \rightarrow \psi \quad &\iff \quad \forall t \subseteq s : t \models \varphi \Rightarrow t \models \psi \\ &\iff \quad \forall t \in \wp(s) : t \models \varphi \Rightarrow t \models \psi \end{aligned}$$

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Mathematical questions involving powersets are often tricky,
not just in set theory (continuum problem) but also in logic.

Example: Medvedev's logic

Consider the class of intuitionistic Kripke frames $\langle \wp(X) - \{\emptyset\}, \supseteq \rangle$ for X finite.
ML is the logic of this class. Is ML axiomatizable/r.e.?

Connection with powerset

The difficulties are brought about by $\rightarrow$, which quantifies over elements of $\wp(s) = \{t \mid t \subseteq s\}$.

$$\begin{aligned} s \models \varphi \rightarrow \psi \quad &\iff \quad \forall t \subseteq s : t \models \varphi \Rightarrow t \models \psi \\ &\iff \quad \forall t \in \wp(s) : t \models \varphi \Rightarrow t \models \psi \end{aligned}$$

Mathematical questions involving powersets are often tricky,
not just in set theory (continuum problem) but also in logic.

Example: Medvedev's logic

Consider the class of intuitionistic Kripke frames $\langle \wp(X) - \{\emptyset\}, \supseteq \rangle$ for X finite.
ML is the logic of this class. Is ML axiomatizable/r.e.? OPEN for >50 years.

Cardinality sentences

- To get a handle on the expressive power of InqBQ,
  it's interesting to look at what it can express about cardinalities.
- This is not interesting for statements ('flat' formulas)
  since then the expressive power coincides with that of FOL.
- But it becomes interesting in connection with cardinality questions.

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Problem

Is the following expressible in InqBQ (possibly wrt finite models)?

(1) How many P are there?

That is, is there a sentence φ such that :

$$s \models \varphi \iff \#P_w \text{ is the same for all } w \in s$$

The inquisitive setting suggests a generalized notion of cardinality quantifier, which covers question-forming quantifiers like 'how many'. We can then ask:

Question

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Using a suitably adapted version of EF games, we can answer this question.

Theorem (Grilletti and C., forthcoming)

A cardinality quantifier is expressible in InqBQ iff it is downward closed and, for some $n \in \mathbb{N}$, it does not distinguish cardinalities above n.

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Corollary
The question 'how many objects are P' is not expressible,
even in restriction to finite models.

Interesting contrast

- What objects are P? $\rightsquigarrow$ expressible
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In view of the importance of 'how many' questions, this suggests a natural extension of InqBQ with the 'how many' quantifier H:

$$s \models Hx.\alpha(x) \iff \text{cardinality of extension of } \alpha(x) \text{ is constant in } s$$

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Task

Study the meta-theoretic properties of the resulting extension InqBQ+H.


THANK
YOU
GRACIAS
ARIGATO
SHUKURIA
GOZAIMASHITA
EFCHARISTO
DANKSCHEEN
JUSPAXAR
KOMAPSUMNIDA
MAAKE
GRAZIE
MEHRBANI
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